Exercise 1 *Specific heat for a 3D electron gas*

The ground state ($T = 0$) energy of a free electron gas is given by $U = \int f(\epsilon)\epsilon D(\epsilon)d\epsilon$ where f is the Fermi-Dirac distribution, $D(\epsilon)$ the density of state and ϵ the energy levels of the electron.

(a) Proof or disproof that $U = \frac{3}{5}\epsilon_F N$ where N is the number of electrons and ϵ_F the Fermi energy.

(b) Using $\int_0^\infty \frac{y^j}{\exp y - y_0 + 1} \approx \frac{y_0^{j+1}}{j+1}\left(1 + \frac{\pi^2 j(j+1)}{6y_0^2}\right)$, show that: $U(T) \approx \frac{V}{2\pi^2}\left(\frac{2m}{\hbar}\right)^{3/2}\frac{\mu^{5/2}}{5/2}\left(1 + \frac{5}{8}\frac{(\pi k_B T)^2}{\mu}\right)$ where μ is the chemical potential and V the volume of the free electron gas.

(c) Use $N = \int_0^\infty f(\epsilon)D(\epsilon)d\epsilon$ to show that $\mu(T) \approx \epsilon_F\left(1 - \frac{\pi^2}{12}\left(\frac{k_B T}{\mu}\right)^2\right)$.

(d) Combine (b) and (c) to show that specific heat is given by $C = \frac{dU}{dT} = \frac{\pi^2}{3}D(\epsilon_F)k_B^2 T$.

Exercise 2 *Electronic specific heat in two dimensions*

Layered crystal structures often have electronic structures that can approximately be considered two-dimensional. The high-temperature superconductor $Tl_2Ba_2CuO_{6+\delta}$ is one such example.

(a) The electronic heat capacity is give by $C_{\text{el}} = \gamma T$. Show that in two dimensions the Sommerfeld parameter γ can be written as $\gamma = \frac{A\pi k_B^2}{3\hbar^2}m$ where m is the electronic mass and A is the total area. What is the unit of C_{el}? Hint: Use $C_{\text{el}} = \frac{1}{3}\pi^2 D(\epsilon_F)k_B^2 T$ and derive the density of state (DOS) in two dimensions.

(b) The crystal structure of $Tl_2Ba_2CuO_{6+\delta}$ consists of stacked layers of CuO_2. Within a layer, the CuO_2 forms a square lattice with a Cu-O lattice distance of $a = 3.8$ Å. The sample area can thus be written as $A = a^2 N$ where N is the number of Cu-O squares. The electronic **specific** heat capacity is measured in units $\text{J mol}^{-1}\,\text{K}^{-1}$. Show that $\gamma = \frac{N_A a^2 \pi k_B^2}{3\hbar^2}m$ where N_A is the Avogadro number.

(c) $Tl_2Ba_2CuO_{6+\delta}$ is observed to have the Sommerfeld parameter $\gamma = 6\,\text{mJ mol}^{-1}\,\text{K}^{-2}$. Using the result of (b), what is the electronic mass m for $Tl_2Ba_2CuO_{6+\delta}$? How does it compare to the free electron mass?

Exercise 3 *Extraction of electronic and phononic specific heat*

In figure 1 the data from a specific heat experiment on Sr_2RuO_4 is shown (adapted from Mackenzie et al. JPSJ **67**, 385 (1998)).

(a) Extract the electronic Sommerfeld parameter γ and the phonon coeffiecient α in $C_{\text{ph}} = \alpha T^3$.

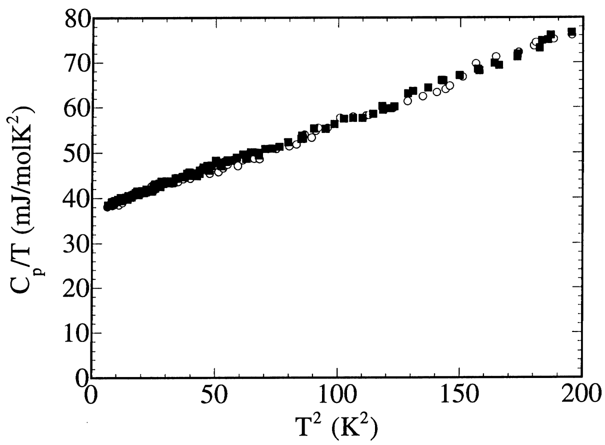

Figure 1: The total specific heat divided by temperature of Sr_2RuO_4 between T_c and 14 K in zero field (filled squares) and a magnetic field of 14 T (open circles) applied parallel to the c-axis.

(b) The crystal structure of Sr_2RuO_4 is similar to the one of $Tl_2Ba_2CuO_{6+\delta}$. Which of the systems would have the larger electronic mass m?

(c) What is the Debye temperature for Sr_2RuO_4?

Exercise 4 *Density-of-States and van-Hove singularities in 2D*

For a square lattice, the simplest tight-binding model for electronic band structure reads:

$$\begin{aligned}\epsilon_{2D}(k_x, k_y) &= -\mu + 2t[\cos(k_x a) + \cos(k_y a)] \\ &+ 4t' \cos(k_x a)\cos(k_y a) + 2t''[\cos(2k_x a) + \cos(2k_y a)],\end{aligned} \tag{1}$$

where t, t', and t'' represent first-, second-, and third- nearest-neighbor hopping parameters, and μ is the chemical potential. The two-dimensional momentum space is spanned by (k_x, k_y). Background information for this exercise can be found in PRL 121, 077004 (2018).

(a) Plot the Fermi surface for half-filling in the cases $t' = t'' = 0$ and $t'/t = -0.5$, $t'' = 0$.

(b) For what (k_x, k_y) does this band structure has a van Hove singularity.

(c) We now define binding energy as $\omega = \epsilon_{2D} - \epsilon_F$ where ϵ_F is the Fermi energy. We are considering the low-temperature limit $k_B T \ll \epsilon_F$. What is the relation between μ and ϵ_F?

(d) The Fermi surface is defined by a closed (k_x, k_y) contour line at $\omega = 0$. We define A as the area enclosed by this contour line. How is A linked to the carrier density?

(e) The notion of enclosed area can be generalized for any ω. Show that $\mathrm{DOS}(\omega) = (a^2/2\pi^2)(dA/d\omega)$.

(f) What happens to the DOS at the ω that host the van Hove singularity?

(g) What would be the expectation for the electronic specific heat if the van Hove singularity is at $\omega = 0$?

(h) What would the effect of disorder? What would happen in a three dimensional system?