Exercise 1. *Weak Decay of the Pion*

1. *Consider the Lagrangian for semileptonic weak interactions:*

$$\mathscr{L} = \frac{4G_F}{\sqrt{2}} \left(\bar{\ell}_L \gamma^\mu \nu_L\right) \left(\bar{u}_L \gamma_\mu d_L\right) + h.c. \tag{1}$$

with $\nu_L = P_L \nu = 1/2(1-\gamma^5)\nu$.
Using the quark currents defined as

$$J^{\mu a} = \bar{Q}\gamma^\mu \tau^a Q \qquad J^{\mu 5a} = \bar{Q}\gamma^\mu\gamma^5\tau^a Q \tag{2}$$

where $Q = (u\ d)^T$ is the quark doublet and $\tau^a = \sigma^a/2$ are the generators of $SU(2)$, show that

$$\bar{u}_L \gamma^\mu d_L = \frac{1}{2}\left(J^{\mu 1} + iJ^{\mu 2} - J^{\mu 51} - iJ^{\mu 52}\right). \tag{3}$$

2. *The matrix element of $J^{\mu 5a}$ between the vacuum and an on-shell pion can be written as*

$$\langle 0|J^{\mu 5a}|\pi^b(p)\rangle = -ip^\mu f_\pi \delta^{ab} e^{-ip\cdot x} \tag{4}$$

where f_π is a constant with the dimension of a mass. Using this identification together with the result of part 1, show that the amplitude for the decay $\pi^+ \to \ell^+\nu$ ($|\pi^+\rangle = 1/\sqrt{2}(|\pi^1\rangle + i\,|\pi^2\rangle)$) is given by

$$\mathscr{M} = G_F f_\pi \bar{u}(q)\not{p}(1-\gamma^5)v(k) \tag{5}$$

where p, k and q are the momenta of the π^+, ℓ^+ and ν respectively.

3. *Compute the decay rate of the pion. Show that this rate vanishes in the limit of zero lepton mass, and that the relative rate of pion decay to muons and electrons is given by*

$$\frac{\Gamma(\pi^+ \to e^+\nu)}{\Gamma(\pi^+ \to \mu^+\nu)} = \left(\frac{m_e}{m_\mu}\right)^2 \frac{\left(1 - m_e^2/m_\pi^2\right)^2}{\left(1 - m_\mu^2/m_\pi^2\right)^2} \approx 10^{-4}. \tag{6}$$

From the measured pion lifetime $\tau_\pi = 2.6 \cdot 10^{-8}$ s, the Fermi constant $G_F = 1.17 \cdot 10^{-5}$ GeV^{-2}, the conversion factor $1\,s = 1.52 \cdot 10^{21}$ MeV^{-1} and the pion and muon masses $m_\pi = 140$ MeV, $m_\mu = 106$ MeV, determine the value of f_π.

Solution.

1. It is useful to define the combinations of $SU(2)$ generators $\tau^\pm = 1/\sqrt{2}(\tau^1 \pm i\tau^2)$,

$$\tau^+ = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \qquad \tau_- = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}, \tag{S.1}$$

to which we can associate the vector and axial-vector isospin currents

$$\begin{aligned} J^{\mu\pm} &= \frac{1}{\sqrt{2}}(J^{\mu 1} \pm iJ^{\mu 2}) = \bar{Q}\gamma^\mu\tau^\pm Q\,,\\ J^{\mu 5\pm} &= \frac{1}{\sqrt{2}}(J^{\mu 51} \pm iJ^{\mu 52}) = \bar{Q}\gamma^\mu\gamma^5\tau^\pm Q\,. \end{aligned} \tag{S.2}$$

With the above definitions, we obtain

$$\begin{aligned}\frac{1}{2}\left(J^{\mu 1}+iJ^{\mu 2}-J^{\mu 51}-iJ^{\mu 52}\right)=&\frac{1}{\sqrt{2}}\left(J^{\mu +}-J^{\mu 5+}\right)\\ =&\frac{1}{\sqrt{2}}\bar{Q}\gamma^\mu(1-\gamma^5)\tau^+Q\\ =&\sqrt{2}\bar{Q}\gamma^\mu P_L\tau^+Q\\ =&\sqrt{2}\bar{Q}_L\gamma^\mu\tau^+Q_L\\ =&(\bar{u}_L,\bar{d}_L)\gamma^\mu\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}u_L\\d_L\end{pmatrix}\\ =&\bar{u}_L\gamma^\mu d_L\,. \end{aligned}\tag{S.3}$$

2. Eq. (S.3) allow us to rewrite the Lagrangian for semileptonic weak interaction as

$$\begin{aligned}\mathscr{L}=&2G_F\left(\bar{\ell}_L\gamma_\mu\nu_L\right)\left(J^{\mu +}-J^{\mu 5+}\right)+\text{h.c.}\\ =&2G_F\left[\left(\bar{\ell}\gamma_\mu P_L\nu\right)\left(J^{\mu +}-J^{\mu 5+}\right)+\left(\bar{\nu}\gamma_\mu P_L\ell\right)\left(J^{\mu -}-J^{\mu 5-}\right)\right]\,,\end{aligned}\tag{S.4}$$

where we used $(J^{\mu\pm})^\dagger=J^{\mu\mp}$ and $(J^{\mu 5\pm})^\dagger=J^{\mu 5\mp}$. In addition, from eq. (4) we have

$$\begin{aligned}\langle 0|J^{\mu 5-}|\pi^+(p)\rangle=&\frac{1}{\sqrt{2}}\left(\langle 0|J^{\mu 51}|\pi^+(p)\rangle-i\langle 0|J^{\mu 52}|\pi^+(p)\rangle\right)\\ =&\frac{1}{2}\left(\langle 0|J^{\mu 51}|\pi^1(p)\rangle+\langle 0|J^{\mu 52}|\pi^2(p)\rangle\right)=-ip^\mu f_\pi e^{-ip\cdot x}\,,\end{aligned}\tag{S.5}$$

and similarly one can verify that $\langle 0|J^{\mu 5+}|\pi^+(p)\rangle=0$.

By making use of eq. (S.5) and of the second term of the interaction Lagrangian in eq. (S.4), we can write the amplitude for the decay $\pi^+\to\ell^+\nu$ as

$$\begin{aligned}i\mathscr{M}(\pi^+\to\ell^+\nu)=&\langle\ell^+(k)\nu(q)|\int d^4x e^{i(k+q)\cdot x}\mathscr{L}|\pi^+(p)\rangle\\ =&-2G_F\bar{u}(q)\gamma_\mu P_L v(k)\int d^4x e^{i(k+q)\cdot x}\langle 0|J^{\mu 5-}|\pi^+(p)\rangle\\ =&2iG_F f_\pi\bar{u}(q)\gamma_\mu P_L v(k)p^\mu\int d^4x e^{i(k+q-p)\cdot x}\\ =&iG_F f_\pi\bar{u}(q)\not{p}(1-\gamma^5)v(k)(2\pi)^4\delta^{(4)}(p-q-k)\,.\end{aligned}\tag{S.6}$$

3. The decay width of the pion is given by

$$\Gamma(\pi^+\to\ell^+\nu)=\frac{1}{2m_\pi}\int d\Phi_2|\mathscr{M}(\pi^+\to\ell^+\nu)|^2\,,\tag{S.7}$$

where the two body-body phase space is defined as

$$\int d\Phi_2=\int\frac{d^4k}{(2\pi)^3}\frac{d^4q}{(2\pi)^3}\delta^+(k^2-m_\ell^2)\delta^+(q^2)(2\pi)^4\delta^4(p-k-q)\,.\tag{S.8}$$

We begin by computing the square of the amplitude derived in 3,

$$
\begin{aligned}
|\mathscr{M}(\pi^+ \to \ell^+\nu)|^2 =& \sum_{\text{spin}} \mathscr{M}^\dagger \mathscr{M} \\
=& G_F^2 f_\pi^2 \sum_{\text{spin}} \left[\bar{u}(q)\not{p}(1-\gamma^5)v(k)\right]^\dagger \left[\bar{u}(q)\not{p}(1-\gamma^5)v(k)\right] \\
=& G_F^2 f_\pi^2 \sum_{\text{spin}} \bar{v}(k)\not{p}(1-\gamma^5)u(q)\bar{u}(q)\not{p}(1-\gamma^5)v(k) \\
=& G_F^2 f_\pi^2 \sum_{\lambda,\lambda'} \bar{v}(k)^\lambda_a[\not{p}(1-\gamma^5)]_{ab}u(q)^{\lambda'}_b \bar{u}(q)^{\lambda'}_c[\not{p}(1-\gamma^5)]_d v(k)^\lambda_d \\
=& G_F^2 f_\pi^2 (\not{k}-m_\ell)_{da}[\not{p}(1-\gamma^5)]_{ab}\not{q}_{bc}[\not{p}(1-\gamma^5)]_d \\
=& G_F^2 f_\pi^2 \mathrm{Tr}[(\not{k}-m_\ell)\not{p}(1-\gamma^5)\not{q}\not{p}(1-\gamma^5)] \\
=& 2G_F^2 f_\pi^2 \mathrm{Tr}[(\not{k}-m_\ell)\not{p}\not{q}\not{p}(1-\gamma^5)]\,,
\end{aligned} \tag{S.9}
$$

where, in the fourth equality, we have made use of the completeness relations

$$
\sum_\lambda v(k)^\lambda \bar{v}(k)^\lambda = (\not{k}-m_\ell)\,, \qquad \sum_{\lambda'} u(q)^{\lambda'}\bar{u}(q)^{\lambda'} = \not{q}\,, \tag{S.10}
$$

and, in the last one, we have used the anticommutation of γ^5, as well as $(1-\gamma^5)^2 = 2(1-\gamma^5)$. We observe that both traces which involve γ^5 give vanishing contribution, since there only two independent momenta and

$$
p_\mu p_\rho \mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] = 4ip_\mu p_\rho \varepsilon^{\mu\nu\rho\sigma} = 0\,. \tag{S.11}
$$

Hence, we are left with

$$
\begin{aligned}
|\mathscr{M}(\pi^+ \to \ell^+\nu)|^2 =& 2G_F^2 f_\pi^2 \mathrm{Tr}[\not{k}\not{p}\not{q}\not{p}] \\
=& 8G_F^2 f_\pi^2 \left((2(k\cdot p)(p\cdot q) - p^2(k\cdot q)\right)\,.
\end{aligned} \tag{S.12}
$$

In a $1 \to 2$ decay, all the invariants can be expressed in terms of masses. In fact, starting from $p = q + k$, $p^2 = m_\pi^2$, $k^2 = m_\ell^2$ and $q^2 = 0$, we have

$$
q\cdot p = \frac{1}{2}(m_\pi^2 - m_\ell^2), \quad k\cdot p = \frac{1}{2}(m_\pi^2 + m_\ell^2), \quad q\cdot k = \frac{1}{2}(m_\pi^2 - m_\ell^2)\,. \tag{S.13}
$$

By inserting these expressions into eq. (S.12) we arrive at

$$
\begin{aligned}
|\mathscr{M}(\pi^+ \to \ell^+\nu)|^2 =& 4G_F^2 f_\pi^2 \left((m_\pi^2 - m_\ell^2)(m_\pi^2 + m_\ell^2) - m_\pi^2(m_\pi^2 - m_\ell^2)\right) \\
=& 4G_F^2 f_\pi^2 m_\pi^2 m_\ell^2 \left(1 - \frac{m_\ell^2}{m_\pi^2}\right)\,.
\end{aligned} \tag{S.14}
$$

We observe that the square matrix element is independent of the four-momenta of the final-state particles. Therefore, we can rewrite eq. (S.7) as

$$
\Gamma(\pi^+ \to \ell^+\nu) = \frac{1}{2m_\pi}|\mathscr{M}(\pi^+ \to \ell^+\nu)|^2\Phi_2 \tag{S.15}
$$

and evaluate the phase space integral independently. The computation is easily carried out in the rest frame of the decaying pion,

$$
p^\mu = (m_\pi, 0, 0, 0)\,. \tag{S.16}
$$

The δ^4 can be used to carry out the integration over k,

$$\begin{aligned}\Phi_2 =& \frac{1}{(2\pi)^2}\int \mathrm{d}^4q\,\delta^+((p-q)^2-m_\ell^2)\delta^+(q^2)\\ =& \frac{1}{(2\pi)^2}\int \mathrm{d}E\mathrm{d}^3\mathbf{q}\,\theta(E)\theta(m_\pi - E)\delta((p-q)^2-m_\ell^2)\delta(q^2)\,,\end{aligned} \tag{S.17}$$

where E is neutrino energy and $\mathbf{q}$ its three-momentum. In addition, we have

$$(p-q)^2 - m_\ell^2 = m_\pi^2 - 2m_\pi E - m_\ell^2\,, \tag{S.18}$$

and, therefore,

$$\delta((p-q)^2-m_\ell^2) = \frac{1}{2m_\pi}\delta\left(E - \frac{m_\pi^2 - m_\ell^2}{2m_\pi}\right). \tag{S.19}$$

This δ-function can be used to integrate over the energy (we note that $E = (m_\pi^2 - m_\ell^2)/(2m_\pi)$ makes both θ-functions redundant) and leads to

$$\begin{aligned}\Phi_2 =& \frac{1}{(2\pi)^2}\frac{1}{2m_\pi}\int \mathrm{d}^3\mathbf{q}\,\delta\left(\left(\frac{m_\pi^2-m_\ell^2}{2m_\pi}\right)^2 - \mathbf{q}^2\right)\\ =& \frac{1}{(2\pi)^2}\frac{1}{4m_\pi}\int \mathrm{d}\mathbf{q}^2(\mathbf{q}^2)^{1/2}\mathrm{d}\Omega_3\delta\left(\left(\frac{m_\pi^2-m_\ell^2}{2m_\pi}\right)^2 - \mathbf{q}^2\right)\\ =& \frac{1}{4\pi}\frac{m_\pi^2-m_\ell^2}{2m_\pi^2} = \frac{1}{8\pi}\left(1-\frac{m_\ell^2}{m_\pi^2}\right)\,,\end{aligned} \tag{S.20}$$

where, in the second equality we have introduced spherical coordinates and, in the third one, we have integrated over the angular variables and used the δ-function to evaluate the leftover integral over $\mathbf{q}^2$.

By inserting this result into eq. (S.15) and by making use of the expression of the square matrix element of eq. (S.14), we arrive at

$$\Gamma(\pi^+ \to \ell^+\nu) = \frac{1}{4\pi}G_F^2 f_\pi^2 m_\pi m_\ell^2\left(1 - \frac{m_\ell^2}{m_\pi^2}\right)^2. \tag{S.21}$$

The ratio of the decay widths into positron and anti-muon is then given by

$$\frac{\Gamma(\pi^+\to e^+\nu)}{\Gamma(\pi^+\to\mu^+\nu)} = \frac{m_e^2}{m_\mu^2}\frac{\left(1-m_e^2/m_\pi^2\right)^2}{\left(1-m_\mu^2/m_\pi^2\right)^2}. \tag{S.22}$$

Finally, if we recall that $\Gamma_\pi \sim 1/\tau_\pi$, we can estimate the value of the pion decay factor f_π as

$$f_\pi = \left(\frac{G_F^2\tau_\pi}{4\pi}m_\pi m_\mu^2\left(1-\frac{m_{mu}^2}{m_\pi^2}\right)^2\right)^{-1/2} \sim 90\,\mathrm{MeV}. \tag{S.23}$$

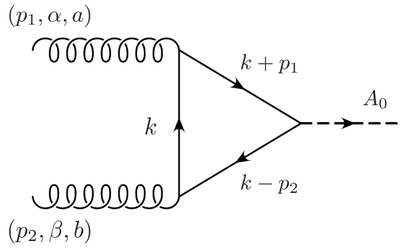

Figure 1: One of the two leading order diagrams for $gg \to A_0$.

Exercise 2. *Pseudoscalar Higgs coupling to gluons*

We consider the production of a pseudoscalar Higgs A_0 in gluon fusion. $gg \to A_0$ is a process induced by a massive quark loop, since A_0 does not couple directly to gluons.

1. *Verify that, at one-loop level, the amplitude can be written as*

$$\mathscr{M} = \varepsilon(p_1)_\alpha \varepsilon(p_2)_\beta \left[\mathscr{M}_1^{\alpha\beta} + \mathscr{M}_2^{\alpha\beta} \right] , \tag{7}$$

where $\mathscr{M}_i^{\alpha\beta}$ are the contributions of two distinct Feynman diagrams. Knowing that the coupling of A_0 to quarks is given by $\frac{-i}{\sqrt{2}} y_f \delta_{ij} \gamma^5$ (where the Yukawa couling $y_f = \sqrt{2} m_f / v$ is proportional to the mass of the fermion m_f and i, j are colour indices) show that the diagram depicted in Figure 1 corresponds to

$$\mathscr{M}_1^{\alpha\beta} = -\frac{y_f (g_s \mu^\epsilon)^2}{\sqrt{2}} \mathrm{Tr}[T^a T^b] \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{Tr}\left[\gamma^5 (\not{k} + \not{p}_1 + m_f) \gamma^\alpha (\not{k} + m_f) \gamma^\beta (\not{k} - \not{p}_2 + m_f)\right]}{((k+p_1)^2 - m_f^2)(k^2 - m_f^2)((k-p_2)^2 - m_f^2)} , \tag{8}$$

where g_s is the coupling of the strong interaction and T^a are colour matrices, which satisfy $\mathrm{Tr}[T^a T^b] = 1/2\ \delta^{ab}$. Evaluate the trace and relate the second diagram with the two gluons exchanged to the first one (Hint: since $\mathscr{M}_1^{\alpha\beta}$ turns out to be finite, you can safely work in four dimensions. Before computing the trace, use the symmetry properties of traces involving γ^5 in order to simplify the calculation).
After perfoming the trace, you will be left with the loop integral

$$I(p_1, -p_2) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{((k+p_1)^2 - m_f^2)(k^2 - m_f^2)((k-p_2)^2 - m_f^2)} = -\frac{i}{8\pi^2} \frac{1}{m_A^2} f(\tau) , \tag{9}$$

with $\tau = \frac{4 m_f^2}{m_A^2}$ and

$$f(\tau) = \begin{cases} \arcsin^2\left(\frac{1}{\sqrt{\tau}}\right) & \tau \geq 1 \\ \frac{-1}{4}\left(\log\left[\frac{1+\sqrt{1-\tau}}{1-\sqrt{1-\tau}}\right] - i\pi\right)^2 & \tau < 1 . \end{cases} \tag{10}$$

2. *Compute the square matrix element and average it over colour and polarisations,*

$$|\overline{\mathscr{M}}|^2 = \frac{1}{N_p^2 N_g^2} \sum_{polarisations} \left|\varepsilon_\alpha(p_1) \varepsilon_\beta(p_2) \mathscr{M}^{\alpha\beta}\right|^2 , \tag{11}$$

where $N_p = 2$ denotes the number of gluon polarisations and $N_g = 8$ is the number of possible values of the colour indices a and b. In the sum over polarisations, you can use

$$\sum_{polarisations} \varepsilon_\nu(k) \varepsilon_\mu^*(k) \to -g_{\mu\nu} ,$$

as in QED. Compute the total partonic cross section $\hat{\sigma}_{gg \to A_0}$.

3. *The total cross section for $pp \to A_0$ can be written as*

$$\sigma_{p\,p\to A_0} = \int \mathrm{d}x_1 \mathrm{d}x_2 g(x_1) g(x_2) \hat{\sigma}_{g\,g\to A_0}\,, \tag{12}$$

where g is the gluon parton distribution function of the proton and x_i are the momentum fractions of the two gluons. By using the result of 2, show that

$$\sigma_{p\,p\to A_0} = \frac{\alpha_s^2}{16\pi}\frac{m_f^4}{v^2 m_A^4}\,|f(\tau)|^2\,\frac{1}{s}\int_{m_A^2/s}^{1}\frac{\mathrm{d}x_1}{x_1} g(x_1) g\left(\frac{m_A^2}{s x_1}\right)\,, \tag{13}$$

where $\alpha_s = g_s^2/(4\pi)$ and s is the total square momentum of the colliding protons.

Solution.

1. At one-loop level the amplitude for $gg \to A_0$ receives contribution from two Feynman diagrams: the one depicted in Figure 1 and the one where the two gluons are exchanged. The first diagram corresponds to

$$\begin{aligned}\mathscr{M}_1^{\alpha\beta} =& (-1)(-ig_s\mu^\epsilon)^2 T^a_{jk}T^b_{ki}\left(-i\frac{y_f}{\sqrt{2}}\delta_{ij}\right)\\ &\times\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\mathrm{Tr}\left[\gamma^5\, i(\not{k}+\not{p}_1+m_f)\gamma^\alpha\, i(\not{k}+m_f)\gamma^\beta\, i(\not{k}-\not{p}_2+m_f)\right]}{((k+p_1)^2-m_f^2)(k^2-m_f^2)((k-p_2)^2-m_f^2)}\\ =& -\frac{y_f(g_s\mu^\epsilon)^2}{\sqrt{2}}\mathrm{Tr}[T^aT^b]\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\mathrm{Tr}\left[\gamma^5(\not{k}+\not{p}_1+m_f)\gamma^\alpha(\not{k}+m_f)\gamma^\beta(\not{k}-\not{p}_2+m_f)\right]}{((k+p_1)^2-m_f^2)(k^2-m_f^2)((k-p_2)^2-m_f^2)}\,.\end{aligned} \tag{S.24}$$

The second diagram can be obtained from the first simply by replacing $(p_1, a, \alpha) \leftrightarrow (p_2, b, \beta)$,

$$\mathscr{M}_2^{\alpha\beta} = -\frac{y_f(g_s\mu^\epsilon)^2}{\sqrt{2}}\mathrm{Tr}[T^aT^b]\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\mathrm{Tr}\left[\gamma^5(\not{k}+\not{p}_2+m_f)\gamma^\beta(\not{k}+m_f)\gamma^\alpha(\not{k}-\not{p}_1+m_f)\right]}{((k+p_2)^2-m_f^2)(k^2-m_f^2)((k-p_1)^2-m_f^2)}\,. \tag{S.25}$$

Let's focus on the trace that appears in eq. (S.24). First, we observe that only terms with γ^5 and four other γ-matrices are non-zero. Secondly, all terms with two $\not{k}$ vanish by parity. Finally, we recall that $\mathrm{Tr}[\gamma^5\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho]$ is antisymmetric under interchange of two adjacent γ-matrices. Hence, the trace is reduced to

$$\begin{aligned}&\mathrm{Tr}\left[\gamma^5(\not{k}+\not{p}_1+m_f)\gamma^\alpha(\not{k}+m_f)\gamma^\beta(\not{k}-\not{p}_2+m_f)\right]\\ &= m_f\,\mathrm{Tr}\left[\gamma^5\left((\not{k}+\not{p}_1)\gamma^\alpha\not{k}\gamma^\beta + (\not{k}+\not{p}_1)\gamma^\alpha\gamma^\beta(\not{k}-\not{p}_2)+\gamma^\alpha\not{k}\gamma^\beta(\not{k}-\not{p}_2)\right)\right]\\ &= m_f\,\mathrm{Tr}\left[\gamma^5(\not{p}_1\gamma^\alpha\not{k}\gamma^\beta+\not{p}_1\gamma^\alpha\gamma^\beta\not{k}-\not{p}_1\gamma^\alpha\gamma^\beta\not{p}_2-\not{k}\gamma^\alpha\gamma^\beta\not{p}_2-\gamma^\alpha\not{k}\gamma^\beta\not{p}_2)\right]\\ &= -m_f\,\mathrm{Tr}\left[\gamma^5\not{p}_1\gamma^\alpha\gamma^\beta\not{p}_2\right] = 4im_f\epsilon^{\mu\nu\alpha\beta}p_{1\mu}p_{2\nu}.\end{aligned} \tag{S.26}$$

This result is symmetric under the exchange $(p_1, \alpha) \leftrightarrow (p_2, \beta)$. Thus, the numerator of $\mathscr{M}_2^{\alpha\beta}$ is the same as the one of $\mathscr{M}_1^{\alpha\beta}$. In addition, the shift $k \to -k$ maps the denominators of the second diagrams to the one of the first one. Therefore, the two diagrams give exactly the same contribution and the amplitude is given by

$$\begin{aligned}\mathscr{M} =&2\,\varepsilon(p_1)_\alpha\varepsilon(p_2)_\beta\,\mathscr{M}_1^{\alpha\beta} = -\frac{iy_f g_s^2}{\sqrt{2}}4m_f\delta^{ab}\epsilon^{\mu\nu\alpha\beta}p_{1\mu}p_{2\nu}\varepsilon(p_1)_\alpha\varepsilon(p_2)_\beta \\ &\times\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{((k+p_1)^2-m_f^2)(k^2-m_f^2)((k-p_2)^2-m_f^2)} \\ =&-\frac{iy_f g_s^2}{\sqrt{2}}4m_f\delta^{ab}\epsilon^{\mu\nu\alpha\beta}p_{1\mu}p_{2\nu}\varepsilon(p_1)_\alpha\varepsilon(p_2)_\beta I(p_1,-p_2)\,.\end{aligned} \tag{S.27}$$

2. Next, we compute the square matrix element and we average it over the degrees of freedom of the initial state, i.e. the two polarisations $N_p = 2$ and the eight possible colour states $N_g = 8$ of each gluon,

$$\begin{aligned}|\,\overline{\mathscr{M}}\,|^2 =&\frac{1}{N_p^2N_g^2}\sum_{\text{polarisations}}\left|\varepsilon_\alpha(p_1)\varepsilon_\beta(p_2)\mathscr{M}^{\alpha\beta}\right|^2 \\ =&\frac{8y_f^2g_s^4}{N_p^2N_g}m_f^2\epsilon^{\mu\nu\alpha\beta}\epsilon_{\rho\sigma\alpha\beta}p_{1\mu}p_{2\nu}p_1^\rho p_2^\sigma\,|I(p_1,-p_2)|^2 \\ =&\frac{16y_f^2g_s^4}{N_p^2N_g}m_f^2(p_1\cdot p_2)^2\,|I(p_1,-p_2)|^2 \\ =&\frac{4y_f^2g_s^4}{N_p^2N_g}m_f^2m_A^4\,|I(p_1,-p_2)|^2 \\ =&\frac{\alpha_s^2}{16\pi^2}\frac{m_f^4}{v^2}\,|f(\tau)|^2\,.\end{aligned} \tag{S.28}$$

In the second equality, we replaced the sum over polarisations with $-g_{\mu\nu}$. In the third equality, we have used $\epsilon^{\mu\nu\alpha\beta}\epsilon_{\rho\sigma\alpha\beta} = -2(\delta^\mu_\rho\delta^\nu_\sigma - \delta^\mu_\sigma\delta^\nu_\rho)$, as well as the mass-shell conditions $p_1^2 = p_2^2 = 0$, and $\hat{s} = 2p_1\cdot p_2 = m_A^2$. We observe that eq. (S.28) is independent of the momentum k of the Higgs. Therefore, we can write the partonic cross section as

$$\begin{aligned}\hat{\sigma}_{gg\to A_0} =&\frac{1}{2\hat{s}}\int\mathrm{d}\Phi_1\,|\,\overline{\mathscr{M}}\,|^2 \\ =&\frac{1}{2\hat{s}}\,|\,\overline{\mathscr{M}}\,|^2\int\frac{\mathrm{d}^4k}{(2\pi)^3}\delta^+(k^2-m_A^2)(2\pi)^4\delta^4(k-p_1-p_2) \\ =&\frac{\pi}{m_A^2}\,|\,\overline{\mathscr{M}}\,|^2\,\delta(\hat{s}-m_A^2) \\ =&\frac{\alpha_s^2}{16\pi}\frac{m_f^4}{v^2m_A^2}\,|f(\tau)|^2\,\delta(\hat{s}-m_A^2)\,,\end{aligned} \tag{S.29}$$

where the leftover δ-function enforces the conservation of the total square momentum.

3. The total cross section for $pp \to A_0$ is obtained from the convolution of $\hat{\sigma}_{gg\to A_0}$ with the gluon parton distribution functions,

$$\begin{aligned}\sigma_{pp\to A_0} =& \int_0^1\mathrm{d}x_1\int_0^1\mathrm{d}x_2\,g(x_1)g(x_2)\,\hat{\sigma}_{gg\to A_0} \\ &\frac{\alpha_s^2}{16\pi}\frac{m_f^4}{v^2m_A^2}\,|f(\tau)|^2\int_0^1\mathrm{d}x_1\int_0^1\mathrm{d}x_2\,g(x_1)g(x_2)\delta(\hat{s}-m_A^2)\,,\end{aligned} \tag{S.30}$$

where, given the momenta P_1 and P_2 of the incoming protons, we have defined $p_i = x_iP_i$. With this definition, the total square momentum of the scattering gluons $\hat{s}$ can be written in terms of the one of the two protons s as $\hat{s} = x_1 x_2\, s$. Hence, we can use the δ-function to evaluate the integral over one of the two momentum fractions x_i,

$$\begin{aligned}\sigma_{pp\to A_0} =& \frac{\alpha_s^2}{16\pi}\frac{m_f^4}{v^2 m_A^2}\,|f(\tau)|^2 \int_0^1 \mathrm{d}x_1 \int_0^1 \mathrm{d}x_2\, g(x_1)g(x_2)\delta(x_1 x_2 s - m_A^2) \\ =& \frac{\alpha_s^2}{16\pi}\frac{m_f^4}{v^2 m_A^2}\,|f(\tau)|^2\, \frac{1}{s}\int_0^1 \frac{\mathrm{d}x_1}{x_1} \int_0^1 \mathrm{d}x_2\, g(x_1)g(x_2)\delta\left(x_2 - \frac{m_A^2}{x_1 s}\right) \\ =& \frac{\alpha_s^2}{16\pi}\frac{m_f^4}{v^2 m_A^2}\,|f(\tau)|^2\, \frac{1}{s}\int_{m_A^2/s}^1 \frac{\mathrm{d}x_1}{x_1}\, g(x_1) g\left(\frac{m_A^2}{s x_1}\right) .\end{aligned} \tag{S.31}$$