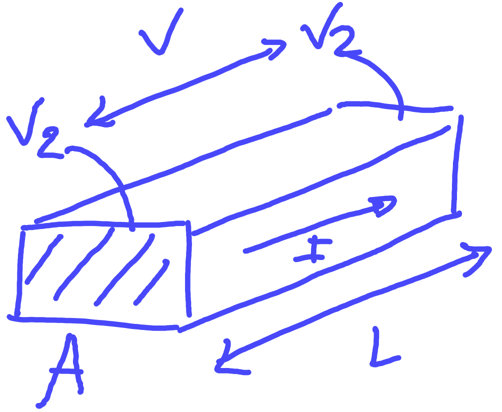
# Free electron model

Electronic Transport

Lecture 3

Kittel Ch. 6

# Recap




Current density : $\vec{j} = -ne\vec{v}$

electron density ($n$)

DRIFT VELOCITY (average e- velocity) ($\vec{v}$)

$\Rightarrow \vec{v} = -\dfrac{e\vec{E}\tau}{m}$ with $\tau$ = relaxation time = average time between 2 collisions

($\tau \sim 10^{-14}$s in metals)

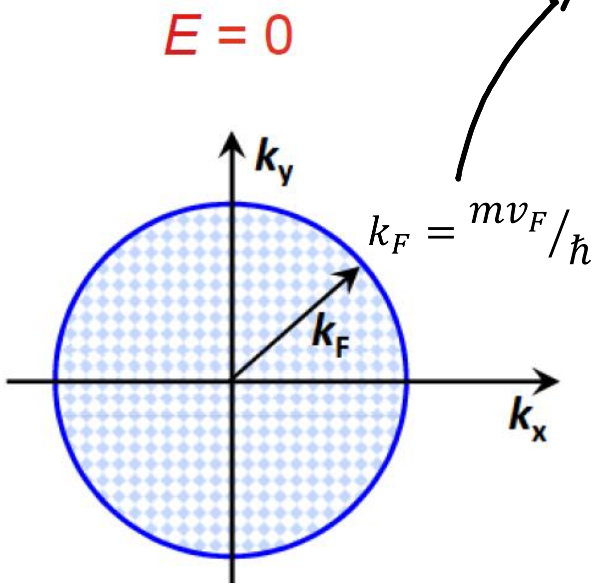
$\Rightarrow \vec{j} = \dfrac{ne^2\tau}{m}\vec{E}$

$\dfrac{ne^2\tau}{m}$ = conductivity $\sigma = \dfrac{1}{\rho}$ ← resistivity ($\rho$)

So far, e- treated "classical".

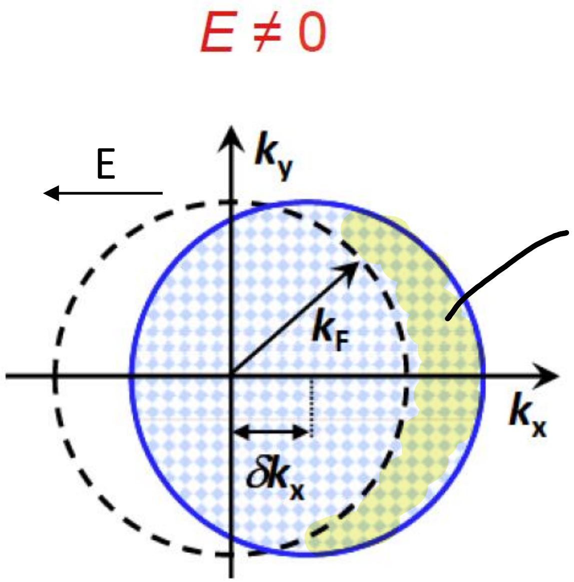
How are these results modified when Q.M. is considered?

$v \propto k$




$k_F = {mv_F}/{\hbar}$

Fermi sphere




uncompensated $e^-$

$E = 0 \Rightarrow$ Fermi sphere centered at 0

total current = 0

all e- move
But for each e- with $\bar{v}$ there is another with velocity $-\bar{v}$

$$\bar{F} = -e\bar{E} = \frac{d\bar{p}}{dt} = \hbar\frac{d\vec{k}}{dt} \rightarrow \delta\vec{k} = \frac{\vec{F}t}{\hbar}$$

- Each e- acquires $\bar{v} = -\frac{e\bar{E}\tau}{m}$
- Fermi spere is displaced (to the right, assuming E to left)

$\Rightarrow$ some e- become now uncompensate $\Rightarrow$ current

The displacement is small because small drift $v$

$E = 0.1\ V/cm \longrightarrow v \sim 1\ cm/s$
(remember $v_F \sim 10^8\ cm/s$)

* estimation of current density (now from Q.M. point of view):

$\frac{v}{v_F}$ fraction of e- which become uncompensate

$n \frac{v}{v_F}$ concentration of e- " " "

e- density

since each e- has velocity $\sim v_F$:

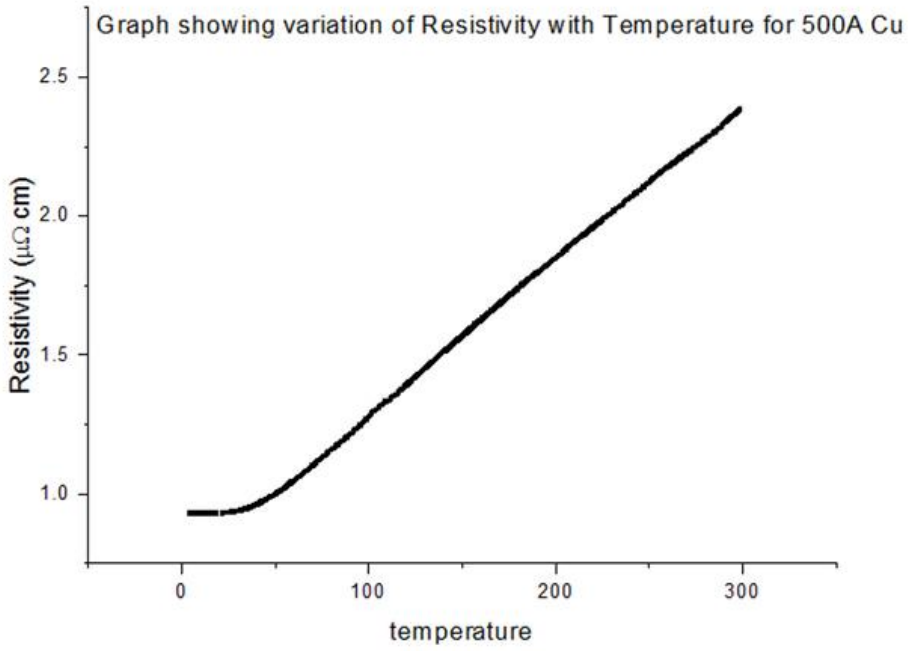
$$\boxed{\vec{j} = -en\left(\frac{\bar{v}}{v_F}\right) \cdot v_F = -en\bar{v}}$$

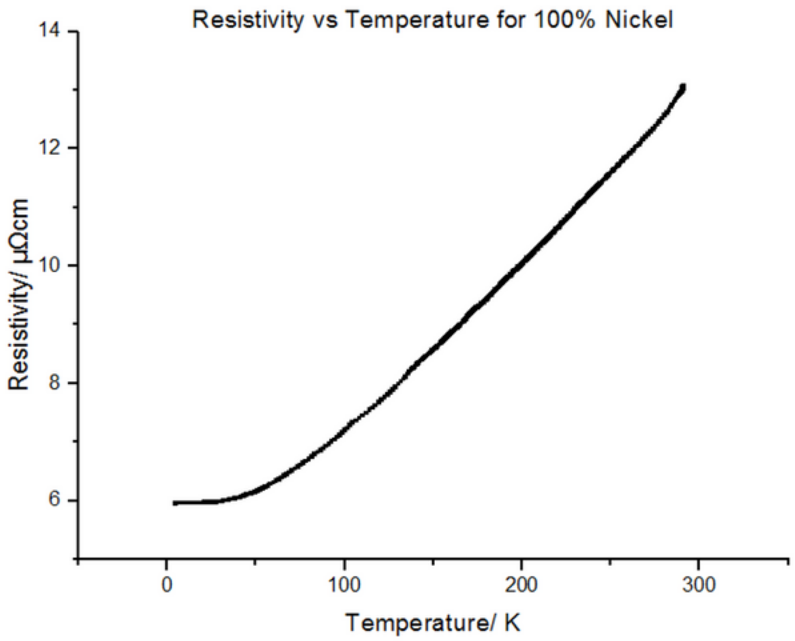
same result found before!
(but remember that "classical" and "quantum" picture are ≠ !!)

since now only e- at Fermi surface contribute to conductance: Mean Free path $l = \tau \cdot v_F$

metal at RT $l \sim 10^{-6}$ cm $= 100$ Å $\sim$ few tens of interatomic distances!

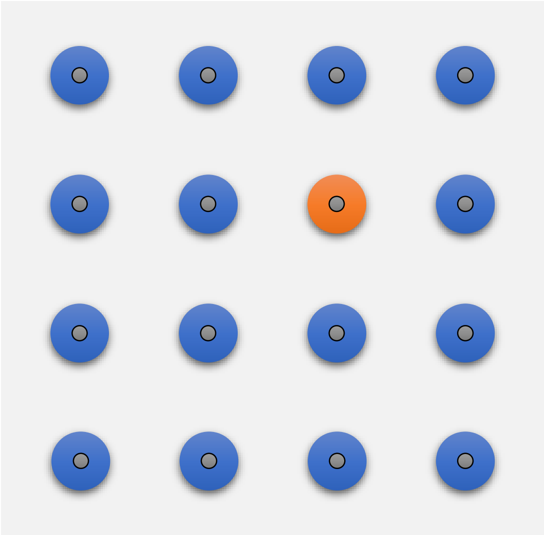
# Electrical resistivity in metals

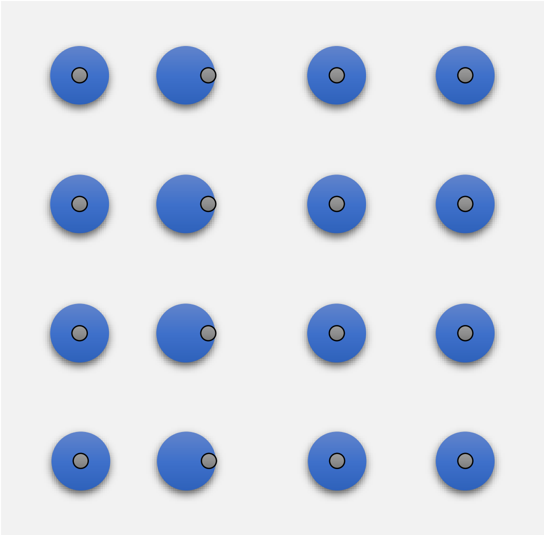



https://steemit.com/physics/@mystifact/let-low-temperature-electron-transport-or-chapter-4-results-and-discussion-part-2-2

# Electrical resistivity in metals





e- undergo collisions becaus Lattice is not perfectly regular:

i) "static" imperfections" (i.e. defects, impurities)

ii) Lattice vibrations (phonons)

$$\rho = \frac{m}{ne^2\tau} \quad ; \quad \frac{1}{\tau}$$ : probability of an electron scattering per unit time

$\tau \sim 10^{-14}s \Rightarrow 10^{14}$ collisions in 1s

$$\Rightarrow \frac{1}{\tau} = \frac{1}{\tau_i} + \frac{1}{\tau_{ph}}$$

$$\boxed{\rho = \rho_i + \rho_{ph}} = \frac{m}{ne^2\tau_i} + \frac{m}{ne^2\tau_{ph}}$$

Matthiessen rule

# Electrical resistivity in metals

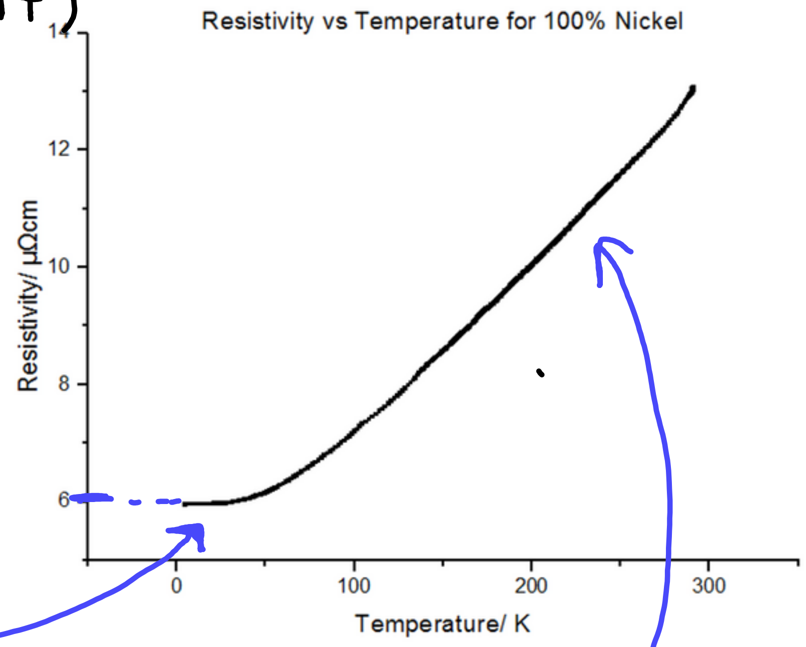
$\rho = \rho_i + \rho_{ph}$

$\rho_i =$ residual resistivity (T-independent)

$\rho_{ph} =$ "lattice resistivity" ($\rho_{ph}(T)$)

at low T,
phonon contribution small
$\Rightarrow \tau_{ph} \to \infty \Rightarrow \rho_{ph} \to 0$

$\rho = \rho_i = ct$

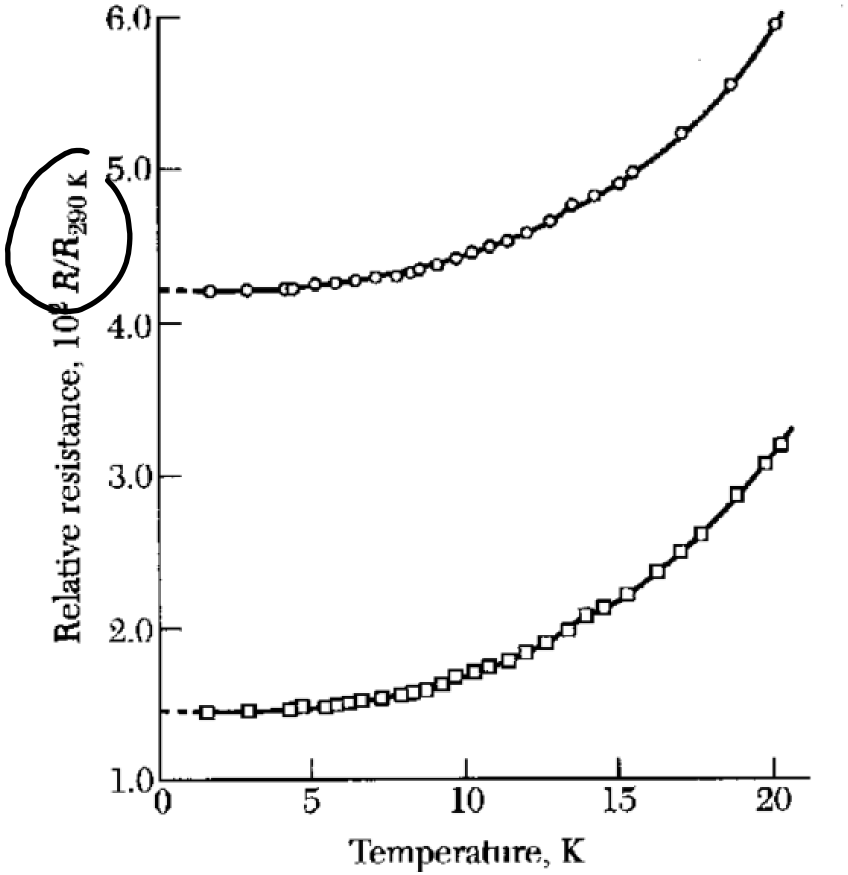



as T↑
phonon contribution↑

Residual resistivity ratio $RRR = \dfrac{\rho(300K)}{\rho(0K)}$

RRR is a measure of purity.

# Example: samples of potassium



# Equation of motion in the presence of an electric field, a magnetic field and scattering

$e^-$ with $\vec{p}$ at time $t$

What is the momentum at time $t+dt$ ?

i) probability $\frac{dt}{\tau}$ that there is a collision before time $t+dt$

ii) probability $(1-dt/\tau)$ that $e^-$ does not scatter : $\vec{p}(t)+\vec{F}.dt$

$$\langle \vec{p}(t+dt) \rangle = \left(1-\frac{dt}{\tau}\right)\left(\vec{p}(t)+\vec{F}dt\right) + 0\,\frac{dt}{\tau}$$

Keeping only linear order terms in $dt$ :

$$\boxed{\frac{d\vec{p}}{dt} = \vec{F} - \frac{\vec{p}}{\tau}}$$

where $\vec{F} = -e(\vec{E}+\vec{v}\times\vec{B})$

if $\left.\begin{matrix} E=0 \\ B=0 \end{matrix}\right\}$ $\vec{p}(t) = \vec{p}_{initial}\, e^{-t/\tau}$

one can think the scattering term $-\vec{p}/\tau$ as just a drag force on

For $F = -e\vec{E} \longrightarrow$ in steady state $\frac{dp}{dt}=0$

$\Longrightarrow$ recover previous results $\vec{J} = \frac{e^2\tau n}{m}\vec{E}$

$$m\left(\frac{d}{dt}+\frac{1}{\tau}\right)\bar{v} = -e\left(\vec{E}+\bar{v}\times\bar{B}\right)$$

consider $\vec{B} = (0, 0, B)$

$$\begin{cases} m\left(\frac{d}{dt}+\frac{1}{\tau}\right)v_x = -e\left(E_x + B v_y\right) \\ m\left(\frac{d}{dt}+\frac{1}{\tau}\right)v_y = -e\left(E_y - B v_x\right) \\ m\left(\frac{d}{dt}+\frac{1}{\tau}\right)v_x = -e E_z \end{cases}$$

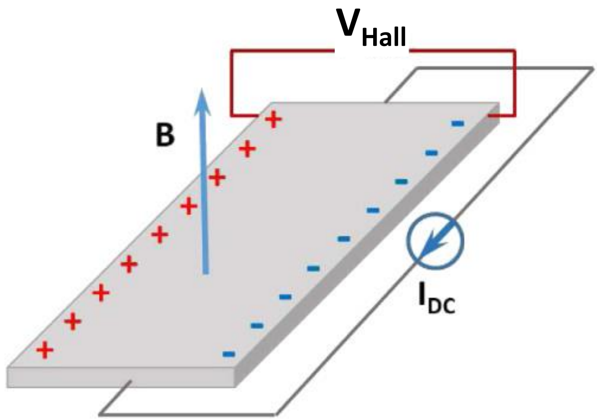
In steady state, $\frac{dv}{dt} = 0 \longrightarrow$ <u>drift velocity</u>:

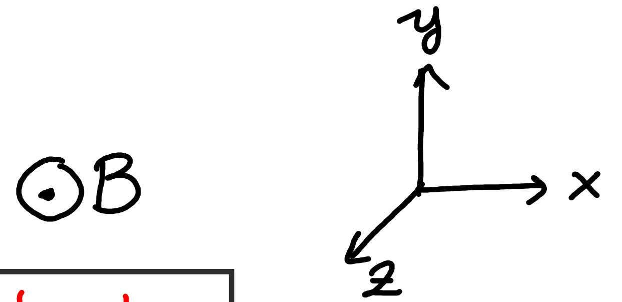
$$\begin{cases} v_x = -\frac{e\cdot\tau}{m}E_x - \omega_c \tau v_y \\ v_y = -\frac{e\tau}{m}E_y + \omega_c\tau v_x \\ v_z = -\frac{e\tau}{m}E_z \end{cases}$$

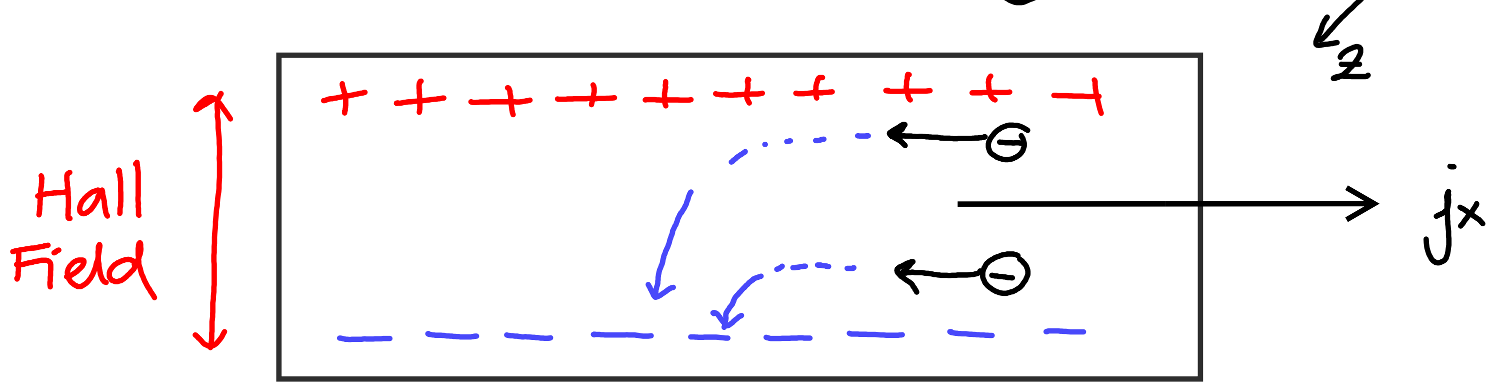
$\omega_c = \frac{eB}{m}$ = cyclotron frequency

# Hall effect

production of a voltage difference across a conductor carrying current in the presence of a magnetic field.
Hall voltage is ⊥ to both current and mag. field




$\delta v_y = 0 \Rightarrow$ transvers electric Field

$$E_y = -\omega_c \tau E_x = -\frac{eB\tau}{m} E_x$$

given $j_x = \frac{ne^2 E_x}{m}$ :

$$E_y = \left(-\frac{1}{ne}\right) j_x B$$

= HALL COEFFICIENT

$$\boxed{R_H = -\frac{1}{ne}}$$

since it only depends on n, Hall Field is used to determine the carrier density

**Comparison of the valence of various atoms to the measured (via Hall resistivity) number of free electrons per atom**

| Metal | Valence | $-1/R_H ne$ |
|---|---|---|
| Li | 1 | 0.8 |
| Na | 1 | 1.2 |
| K | 1 | 1.1 |
| Rb | 1 | 1.0 |
| Cs | 1 | 0.9 |
| Cu | 1 | 1.5 |
| Ag | 1 | 1.3 |
| Au | 1 | 1.5 |
| Be | 2 | -0.2 |
| Mg | 2 | -0.4 |
| In | 3 | -0.3 |
| Al | 3 | -0.3 |

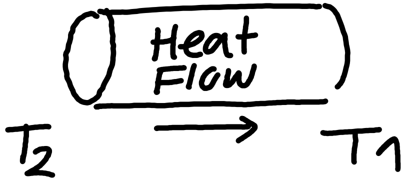
Very good agreement between measurements and calculation of 1 conduction e- per atom

strong disagreement!
And even the sign is opposite to the expected one for a negatively charged e⁻ !!

This result will be explained using band theory of sol

$$R_H = \frac{E_y}{j_x B_z} = -\frac{1}{|e| n_e}$$

# Thermal conductivity of metals




$T_2 > T_1$

Heat current density $\quad J_q = -\kappa \dfrac{dT}{dx}$

thermal conductivity

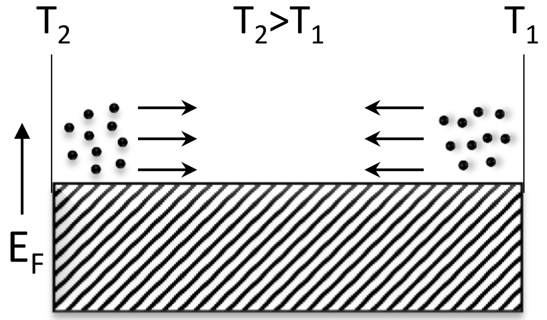
Insulator: heat carried by phonons

Metals: heat carried by both phonons and electrons

$$\kappa = \kappa_e + \kappa_{ph}$$

For a pure metal, $\kappa_e > \kappa_{ph}$

(often $\kappa_e \sim 10^2 \kappa_{ph}$)



previously derived: $\kappa = \frac{1}{3} C_{el} v_F l$

$C_{el}$ = Electronic Heat Capacity
$v_F$ = Fermi velocity
$l$ = mean free path of $e^-$ at $E_F$

$$C_{el} = \frac{\pi^2}{2} N K_B^2 \frac{T}{E_F}$$

$$\kappa = \frac{1}{3}\left(\frac{\pi^2}{2} N K_B^2 \frac{T}{E_F}\right) \cdot v_F l = \frac{\pi^2 n K_B^2 \tau T}{3m}$$

$E_F = \frac{1}{2} m v_F^2$

$l = \tau \cdot v_F$

$$\sigma = \frac{n e^2 \tau}{m}$$

$$\boxed{\frac{\kappa}{\sigma} = \frac{1}{3}\left(\frac{\pi K_B}{e}\right)^2 T = L T}$$

Wiedemann-Franz Law

Lorentz number $= 2.45 \cdot 10^{-8} \frac{W\Omega}{K^2}$ = ct independent of metal

electrical and conductivity are intimately related (expected since both carried by $e^-$)

**Table 5 Experimental Lorenz numbers**

| | $L \times 10^8$ watt-ohm/deg$^2$ | | | $L \times 10^8$ watt-ohm/deg$^2$ | |
|---|---|---|---|---|---|
| Metal | 0°C | 100°C | Metal | 0°C | 100°C |
| Ag | 2.31 | 2.37 | Pb | 2.47 | 2.56 |
| Au | 2.35 | 2.40 | Pt | 2.51 | 2.60 |
| Cd | 2.42 | 2.43 | Su | 2.52 | 2.49 |
| Cu | 2.23 | 2.33 | W | 3.04 | 3.20 |
| Mo | 2.61 | 2.79 | Zn | 2.31 | 2.33 |

value predicted is fairly close to the experimental one.

# Summary

➢ **Free electron model is the simplest way to describe metals and gives a good insight into properties such as**

- Heat capacity $C_{el} = \gamma T$
- Thermal conductivity
- Electric conductivity

* Fermi Energy $E_F$ (= chemical potential at 0K)

* Electronic Transport : collision time $\tau$, conductivity $\sigma = ne^2\tau/m$ ($\vec{j} = \sigma\vec{E}$), resistivity $\rho = 1/\sigma$, Hall coefficient $R_H = -\frac{1}{ne}$ (measures density of $e^-$)

Fermi wavevector, Fermi velocity, Fermi sphere...