MMP 2
Solutions 3

FS 19
Prof. Ph. Jetzer

Exercise 1

a) When $f'(z_0) = \frac{df}{dz}\big|_{z_0} = \lim\limits_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists.

b) f is holomorphic at z_0 (or everywhere) when it is differentiable at z_0 (or everywhere)

c) f is analytic at z_0 (or everywhere) when it is differentiable at z_0 (or everywhere)

d) They satisfy the Cauchy-Riemann equations:

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\
\frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \\
f'(z) = \frac{\partial f}{\partial x} &= \frac{1}{i}\frac{\partial f}{\partial y} = u_x - iu_y = v_y + iv_x
\end{aligned}$$

Exercise 2 $w = u + iv$

a) i)

$$\begin{aligned}
u &= 3x^2y - y^3 \\
\frac{\partial v}{\partial y} &= \frac{\partial u}{\partial x} = 6xy && \Rightarrow v = 3xy^2 + f(x) \\
\frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y} = -3x^2 + 3y^3 && \Rightarrow v = -x^3 + 3xy^2 + g(y) \\
\Rightarrow v(x,y) &= 3xy^2 - x^3 + C
\end{aligned}$$

ii)

$$\begin{aligned}
v &= e^{-x}\cos y \\
\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} = -e^{-x}\sin y && \Rightarrow u = e^{-x}\sin y + f(y) \\
\frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} = +e^{-x}\cos y && \Rightarrow u = e^{-x}\sin y + g(x) \\
\Rightarrow u &= e^{-x}\sin y + C
\end{aligned}$$

b) i) No:

$$
\begin{aligned}
f(z) &= |z| = \sqrt{x^2+y^2} \\
\frac{\partial f}{\partial x} &= (x^2+y^2)^{-\frac{1}{2}} x \neq (x^2+y^2)^{-\frac{1}{2}} \frac{y}{i} = \frac{1}{i}\frac{\partial f}{\partial y}
\end{aligned}
$$

ii) No:

$$
\begin{aligned}
f(z) &= \bar{z} = x - iy \\
i\frac{\partial f}{\partial x} &= i \cdot 1 = i \neq -i = \frac{\partial f}{\partial y}
\end{aligned}
$$

iii) Yes:

$$
\begin{aligned}
f(z) &= e^z = e^x e^{iy} \\
\frac{\partial f}{\partial x} &= e^x e^{iy} \\
\frac{\partial f}{\partial y} &= ie^x e^{iy} \\
\Rightarrow f_x &= \frac{1}{i} f_y
\end{aligned}
$$

iv) Yes:

$$
\begin{aligned}
f(x) &= \cos(z) = \frac{1}{2}\left(e^{iz} + e^{-iz}\right) \\
\frac{\partial f}{\partial x} &= \frac{1}{2}\left(e^{iz} - e^{-iz}\right) \\
\frac{\partial f}{\partial y} &= \frac{i}{2}\left(e^{iz} - e^{-iz}\right) \\
\Rightarrow f_x &= \frac{1}{i} f_y
\end{aligned}
$$

Exercise 3

$$
\begin{aligned}
\left(\frac{ia-1}{ia+1}\right)^{ib} &= \exp\left[ib \ln\left(\frac{ia-1}{ia+1}\right)\right] \\
\operatorname{arccot}(a) &= \frac{i}{2}\left[\ln\left(\frac{a-i}{a}\right) - \ln\left(\frac{a+i}{a}\right)\right] \\
&= \frac{i}{2}\ln\left(\frac{a-i}{a+i}\right) \\
&= \frac{i}{2}\ln\left(\frac{ia+1}{ia-1}\right) \\
&= -\frac{i}{2}\ln\left(\frac{ia-1}{ia+1}\right) \\
\Rightarrow \exp(-2b \cdot \operatorname{arccot}(a)) &= \exp\left[-2b \cdot \left(-\frac{i}{2}\right)\ln\left(\frac{ia-1}{ia+1}\right)\right] \\
&= \exp\left[ib \ln\left(\frac{ia-1}{ia+1}\right)\right] \\
&= \left(\frac{ia-1}{ia+1}\right)^{ib}
\end{aligned}
$$

Also:

$$\cot z = \frac{i(e^{iz} + e^{-iz})}{e^{iz} - e^{-iz}}$$

Exercise 4 Solve $e^z = w$

$$\arg z = \begin{cases} \arctan \frac{y}{x} & x > 0 \\ \arctan \frac{y}{x} + \pi & x < 0, y \geq 0 \\ \arctan \frac{y}{x} - \pi & x < 0, y < 0 \end{cases}$$

$$-\pi < \arg z \leq \pi$$

$$\arg z = \theta$$

$$\ln z = \ln r + i(\theta + 2nr) \qquad N \in \mathbb{Z}$$

a)

$$w = -1 \qquad \Rightarrow x = -1, y = 0$$

$$r = 1, \theta = \pi$$

$$\Rightarrow z = \pi i(1 + 2n)$$

b)

$$w = \frac{-1 - i}{\sqrt{2}} \qquad \Rightarrow x = -\frac{1}{\sqrt{2}}, y = -\frac{1}{\sqrt{2}}$$

$$r = 1, \theta = \arctan(1) - \pi = -\frac{3}{4}\pi$$

$$\Rightarrow z = \pi i(-\frac{3}{4} + 2n)$$

c)

$$w = 4 - 4i \qquad \Rightarrow x = 4, y = -4$$

$$\Rightarrow r = 4\sqrt{2}, \theta = -\frac{\pi}{4}$$

$$\Rightarrow z = \ln(4\sqrt{2}) + i\pi(-\frac{1}{4} + 2n)$$

d)

$$w = -3 + \sqrt{3}i \qquad \Rightarrow x = -3, y = \sqrt{3}$$

$$r = 2\sqrt{3}, \theta = \frac{5}{6}\pi$$

$$\Rightarrow z = \ln(2\sqrt{3}) + i\pi(\frac{5}{6} + 2n)$$

Exercise 5

a)

$$\begin{aligned}\sin z &= \sin(x+iy)\\ &= \sin(x)\cos(iy)+\cos(x)\sin(iy)\\ &= \sin(x)\cosh(y)+i\cos(x)\sinh(y)\end{aligned}$$

$$\begin{aligned}\cosh(y) &\overset{y\in\mathbb{R}}{\geq} 1 \Rightarrow x = n\pi, n\in\mathbb{Z}\\ \cos(n\pi) &\neq 0 \Rightarrow y = 0\\ &\Rightarrow z = n\pi\end{aligned}$$

$$\begin{aligned}\cosh(z) &= \cosh(x+iy)\\ &= \cosh(x)\cosh(iy)+\sinh(x)\sinh(iy)\\ &= \cosh(x)\cos(y)+i\sinh(x)\sin(y)\\ \Rightarrow y &= (2n+1)\frac{\pi}{2}, n\in\mathbb{Z}, x=0\\ z &= i(2n+1)\frac{\pi}{2}\end{aligned}$$

b)

$$\begin{aligned}i &= \cos z\\ &= \frac{1}{2}(e^{iz}+e^{-iz})\\ ie^{iz} &= \frac{1}{2}(e^{2iz}+1)\\ 0 &= e^{2iz}-2ie^{iz}+1\\ \Rightarrow e^{iz} &= \frac{2i\pm\sqrt{-4-4}}{2}\\ &= i(1\pm\sqrt{2})\\ x &= 0\\ y &= (1\pm\sqrt{2})\\ \Rightarrow r &= \pm 1+\sqrt{2}\\ \theta &= \pm\frac{\pi}{2}\\ \Rightarrow iz &= \ln(\sqrt{2}\pm 1)+i(\pm\frac{\pi}{2}+2n\pi), n\in\mathbb{Z}\\ z &= -i\ln(\sqrt{2}\pm 1)+\pi(2n\pm\frac{1}{2})\end{aligned}$$

c) Find $\lim\limits_{|z|\to\infty} \frac{\sin z}{z} = f(z)$

$$z = x: \qquad f(z) = \lim_{x\to\infty} \frac{\sin x}{x} = 0$$

$$z = iy: \qquad f(z) = \lim_{y\to\infty} \frac{\sin iy}{iy} = \infty$$

$$\lim_{y\to\infty} \frac{\sin iy}{iy} = \lim_{y\to\infty} \frac{\sinh y}{iy} \overset{\text{De l'Hopital}}{=} \lim_{y\to\infty} \frac{\cosh}{i} = \infty$$

Different values $\Rightarrow \lim\limits_{|z|\to\infty} \frac{\sin(x+iy)}{x+iy}$ does not exist.