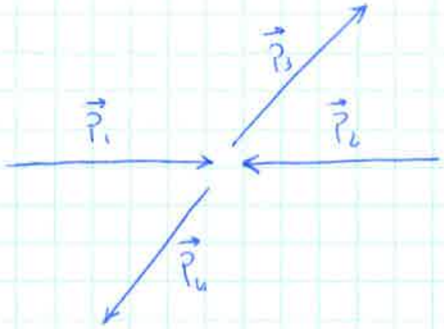
KT II - Exercise sheet 1

Two body scattering

a) Derive the following equation in the centre of mass frame for a 2-body scattering

$$\sqrt{(P_1 \cdot P_2)^2 - (m_1 m_2)^2} = (E_1 + E_2)\,|\vec{P}_1|$$



CM frame $\Rightarrow$ $\vec{P}_1 = -\vec{P}_2$

$P_1 = (E_1, \vec{P}_1)$

$P_2 = (E_2, -\vec{P}_1)$

- $P_1 \cdot P_2 = E_1 E_2 - \vec{P}_1 \cdot |\vec{P}_1| = E_1 E_2 + \vec{P}_1^{\,2}$
- $(P_1 \cdot P_2)^2 - (m_1 m_2)^2 = (E_1 E_2 + \vec{P}_1^{\,2})^2 - (m_1 m_2)^2$

$$= \quad = E_1^2 E_2^2 + 2E_1E_2 |\vec{P}_1|^2 + |\vec{P}_1|^4 - m_1^2 m_2^2$$

But $m_1^2 = E_1^2 - \vec{P}_1^{\,2}$ and $m_2^2 = E_2^2 - \vec{P}_2^{\,2} = E_2^2 - \vec{P}_1^{\,2}$

$$\Rightarrow (P_1 \cdot P_2)^2 - (m_1 m_2)^2 = E_1 E_2^2 + 2E_1E_2|\vec{P}_1|^2 + |\vec{P}_1|^4 - (E_1^2 - |\vec{P}_1|^2)(E_2^2 - |\vec{P}_1|^2)$$

$$= \cancel{E_1 E_2^2} + 2E_1E_2|\vec{P}_1|^2 + \cancel{|\vec{P}_1|^4} - \cancel{E_1 E_2^2} + E_2^2|\vec{P}_1|^2 + E_1^2|\vec{P}_1|^2 - \cancel{|\vec{P}_1|^4}$$

$$= |\vec{P}_1|^2 (2E_1E_2 + E_2^2 + E_1^2) = |\vec{P}_1|^2 (E_1 + E_2)^2$$

$$\Rightarrow \sqrt{(P_1 \cdot P_2)^2 - (m_1 m_2)^2} = (E_1 + E_2)\,|\vec{P}_1|$$

b) Derive the same formula in the lab frame (target at rest)

In the lab frame $P_1 = (E_1, \vec{P}_1)$ and $P_2 = (m_2, 0)$

- $P_1 \cdot P_2 = E_1 \cdot m_2$
- $(P_1 \cdot P_2)^2 - (m_1 m_2)^2 = E_1^2 m_2^2 - m_1^2 m_2^2 = m_2^2 (E_1^2 - m_1^2) = m_2^2 |\vec{P}_1|^2$

$$\Rightarrow \sqrt{(P_1 \cdot P_2)^2 - (m_1 m_2)^2} = |\vec{P}_1|\, m_2$$

Pauli Matrices

a) $[\sigma_x, \sigma_y] = 2i\sigma_z$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$\sigma_x \sigma_y - \sigma_y \sigma_x =$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = 2i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 2i\sigma_z$$

Similarly for $[\sigma_y, \sigma_z] = 2i\sigma_x$ and $[\sigma_z, \sigma_x] = 2i\sigma_y$

d) $[\sigma^2, \sigma_i] = 0$

$$\sigma^2 = \sigma_x^2 + \sigma_y^2 + \sigma_z^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

$$[\sigma^2, \sigma_x] = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix} = 0$$

The same for σ_y and σ_z because σ^2 is diagonal.

Dirac hamiltonian

$$H = \vec{\alpha}\cdot\vec{p} + \beta m$$

a) $[H, \gamma^5]$ What does it happen if the particle is massless

$$[H,\gamma^5] = [\vec{\alpha}\vec{p} + \beta m, \gamma^5] = \begin{pmatrix} m & \vec{\sigma}\vec{p} \\ \vec{\sigma}\vec{p} & -m \end{pmatrix}\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} - \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}\begin{pmatrix} m & \vec{\sigma}\vec{p} \\ \vec{\sigma}\vec{p} & -m \end{pmatrix}$$

$$= \begin{pmatrix} \vec{\sigma}\cdot\vec{p} & m \\ -m & \vec{\sigma}\cdot\vec{p} \end{pmatrix} - \begin{pmatrix} \vec{\sigma}\cdot\vec{p} & -m \\ m & \vec{\sigma}\vec{p} \end{pmatrix} = +2m \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} = 0 \text{ only if } m=0$$

2) DIRAC EQUATION

a) $(\not{p} - m)\, u(p) = 0 \quad , \quad u(-p) = v(p)$

$(-\not{p} - m)\, u(-p) = 0 \Rightarrow (\not{p} + m)\, v(p) = 0$

b) $\bar{u} = u^{+}\gamma^{0}$

$(\gamma^{\mu} p_{\mu} - m)\, u = 0 \Rightarrow \left((\gamma^{\mu} p_{\mu} - m)\, u\right)^{+} = 0 \Rightarrow u^{+}\gamma^{\mu +} p_{\mu} - m\, u^{+} = 0$

MULTIPLYING BY γ^{0} WE OBTAIN:

$u^{+}\gamma^{\mu +}\gamma^{0} p_{\mu} - m\, u^{+}\gamma^{0} = 0$ BUT $\gamma^{\mu +}\gamma^{0} = \gamma^{0}\gamma^{\mu}$

$\Rightarrow u^{+}\gamma^{0}\gamma^{\mu} p_{\mu} - m\, u^{+}\gamma^{0} = 0 \Rightarrow \overline{u(p)}\,(\not{p} - m) = 0$

LIKEWISE FOR THE ANTIPARTICLE SPINOR $\bar{v} \Rightarrow \overline{v(p)}\,(\not{p} + m) = 0$

c) $u^{(1)} = N \begin{pmatrix} 1 \\ 0 \\ \frac{c p_z}{E + mc^2} \\ \frac{c(p_x + i p_y)}{E + mc^2} \end{pmatrix} , \quad u^{(2)} = N \begin{pmatrix} 0 \\ 1 \\ \frac{c(p_x - i p_y)}{E + mc^2} \\ \frac{-c p_z}{E + mc^2} \end{pmatrix}$

$$u^{(1)+} u^{(2)} = N^{*} \begin{pmatrix} 1 & 0 & \frac{c p_z}{E + mc^2} & \frac{c(p_x - i p_y)}{E + mc^2} \end{pmatrix} N \begin{pmatrix} 0 \\ 1 \\ \frac{c(p_x - i p_y)}{E + mc^2} \\ \frac{-c p_z}{E + mc^2} \end{pmatrix} =$$

$$= |N|^2 \left(\frac{c^2 p_z (p_x - i p_y)}{E + mc^2} - \frac{c^2 p_z (p_x - i p_y)}{E + mc^2} \right) = 0$$

$$v^{(1)} = N \begin{pmatrix} \frac{c(P_x - iP_y)}{E+mc^2} \\ \frac{c(-P_z)}{E+mc^2} \\ 0 \\ 1 \end{pmatrix} , \quad v^{(2)} = N \begin{pmatrix} \frac{c(P_z)}{E+mc^2} \\ \frac{c(P_x+iP_y)}{E+mc^2} \\ 1 \\ 0 \end{pmatrix}$$

$$v^{(1)\dagger} v^{(2)} = |N|^2 \left(\frac{c^2 P_z(P_x+iP_y)}{(E+mc^2)^2} - \frac{c^2 P_z(P_x+iP_y)}{(E+mc^2)^2} \right) = 0$$

d) Normalization factors

$\bar{u}u = 2mc \quad , \bar{v}v = -2mc$

$$\bar{u}u = u^\dagger \gamma^0 u = |N|^2 (u_A^\dagger \; u_B^\dagger) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} u_A \\ u_B \end{pmatrix} = |N|^2 (u_A^\dagger u_A - u_B^\dagger u_B)$$

$$u_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad u_B = \frac{c}{E+mc^2} \begin{pmatrix} P_z \\ P_x + iP_y \end{pmatrix}$$

$$u_A^\dagger u_A = (1 \quad 0) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1$$

$$u_B^\dagger u_B = \left(\frac{c}{E+mc^2} \right)^2 (P_z \quad P_x - iP_y) \begin{pmatrix} P_z \\ P_x + iP_y \end{pmatrix} = \left(\frac{c}{E+mc^2} \right)^2 (P_z^2 + P_x^2 + P_y^2)$$

$$\bar{u}u = |N|^2 (u_A^\dagger u_A - u_B^\dagger u_B) = |N|^2 \left(\frac{c}{E+mc^2} \right)^2 |P|^2$$

where $|N|$ is the normalizion factor $|N|^2 = \frac{E+mc^2}{c}$

$$\bar{u}u = \frac{E+mc^2}{c} \left(1 - \frac{E^2 - m^2c^4}{(E+mc^2)^2} \right) = \frac{(E+mc^2)}{c} \frac{-(E-mc^2)(E+mc^2) + (E+mc^2)^2}{(E+mc^2)^2} =$$

$$= \frac{3mc^2}{c} = 2mc$$

LIKEWISE FOR ANTIPARTICLE SPINORS

$$\bar{v}v = v^\dagger \gamma^0 v = |N|^2 (v_A^\dagger \quad v_B^\dagger) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} v_A \\ v_B \end{pmatrix} = |N|^2 (v_A^\dagger v_A - v_B^\dagger v_B)$$

$$v_B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad v_A = \frac{c}{E + mc^2} \begin{pmatrix} p_x - ip_y \\ -p_z \end{pmatrix}$$

$$\bar{v}v = |N|^2 \left[\left(\frac{c}{E + mc^2} \right)^2 \begin{pmatrix} p_x + ip_y & -p_z \end{pmatrix} \begin{pmatrix} p_x - ip_y \\ -p_z \end{pmatrix} - (0 \quad 1) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] =$$

$$= \frac{E + mc^2}{c} \left[\left(\frac{c}{E + mc^2} \right)^2 (p_x^2 + p_y^2 + p_z^2) - 1 \right] =$$

$$= \frac{E + mc^2}{c} \left[\frac{c^2 p^2 - (E + mc^2)^2}{(E + mc^2)^2} \right] = \frac{1}{c} \left[\frac{(E - mc^2)(E + mc^2) - (E + mc^2)^2}{E + mc^2} \right] =$$

$$= -2mc$$

e) Verify completeness relations:

$$\sum_{s=1,2} u^{(s)} \bar{u}^{(s)} = (\gamma^\mu p_\mu + mc), \quad \sum_{s=1,2} v^{(s)} \bar{v}^{(s)} = (\gamma^\mu p_\mu - mc)$$

$$\sum_{s=1,2} u^{(s)} \bar{u}^{(s)} = \left\{ |N|^2 \begin{pmatrix} 1 \\ 0 \\ \frac{cp_z}{E + mc^2} \\ \frac{c(p_x + ip_y)}{E + mc^2} \end{pmatrix} \begin{pmatrix} 1 & 0 & -\frac{cp_z}{E + mc^2} & \frac{-c(p_x - ip_y)}{E + mc^2} \end{pmatrix} \right\} +$$

$$\left\{ |N|^2 \begin{pmatrix} 0 \\ 1 \\ \frac{c(p_x - ip_y)}{E + mc^2} \\ \frac{-cp_z}{E + mc^2} \end{pmatrix} \begin{pmatrix} 0 & 1 & -\frac{c(p_x + ip_y)}{E + mc^2} & \frac{+cp_z}{E + mc^2} \end{pmatrix} \right\}$$

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$$= \left(\frac{E+mc^2}{c}\right)\left\{\begin{pmatrix} 1 & 0 & -\frac{cP_z}{E+mc^2} & -\frac{c(P_x-iP_y)}{E+mc^2} \\ 0 & 0 & 0 & 0 \\ \frac{cP_z}{E+mc^2} & 0 & -\frac{c^2P_z^2}{(E+mc^2)^2} & -\frac{c^2P_z(P_x-iP_y)}{(E+mc^2)^2} \\ \frac{c(P_x+iP_y)}{E+mc^2} & 0 & -\frac{c^2P_z(P_x+iP_y)}{(E+mc^2)^2} & -\frac{c^2(P_x^2+P_y^2)}{(E+mc^2)^2} \end{pmatrix} + \right.$$

$$\left. \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -\frac{c(P_x+iP_y)}{E+mc^2} & \frac{cP_z}{E+mc^2} \\ 0 & \frac{c(P_x-iP_y)}{E+mc^2} & -\frac{c^2(P_x^2+P_y^2)}{(E+mc^2)^2} & \frac{c^2P_z(P_x-iP_y)}{(E+mc^2)^2} \\ 0 & -\frac{cP_z}{E+mc^2} & \frac{c^2P_z(P_x+iP_y)}{(E+mc^2)^2} & -\frac{c^2P_z^2}{E+mc^2} \end{pmatrix}\right\}$$

$$= \begin{pmatrix} \frac{E+mc^2}{c} & 0 & -P_z & -(P_x-iP_y) \\ 0 & \frac{E+mc^2}{c} & -(P_x+iP_y) & P_z \\ P_z & P_x-iP_y & -\frac{c|\vec{P}|^2}{E+mc^2} & 0 \\ P_x+iP_y & -P_z & 0 & -\frac{c|\vec{P}|^2}{E+mc^2} \end{pmatrix} =$$

$$\boxed{\begin{array}{l} \text{But } \vec{p}\,\vec{\sigma} = \begin{pmatrix} P_z & P_x-iP_y \\ P_x+iP_y & -P_z \end{pmatrix} \\ \frac{c^2|\vec{P}|^2}{E+mc^2} = \frac{E^2-m^2c^4}{E+mc^2} = E-mc^2 \end{array}}$$

$$= \begin{pmatrix} \begin{pmatrix} \frac{E+mc^2}{c} & 0 \\ 0 & \frac{E+mc^2}{c} \end{pmatrix} & -\vec{p}\cdot\vec{\sigma} \\ \vec{p}\cdot\vec{\sigma} & \begin{pmatrix} -\frac{E-mc^2}{c} & 0 \\ 0 & -\frac{E-mc^2}{c} \end{pmatrix} \end{pmatrix} = \frac{E}{c}\gamma^0 + mc - \vec{p}\begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} =$$

$$= \gamma^\mu P_\mu + mc$$

C.V.D

4) COULOMB SCATTERING

Our electromagnetic field is described by the potential:

$A_\mu = (V, \vec{A}) = (V, 0)$ where $V = \frac{Ze}{4\pi|\vec{x}|}$ $\rightarrow$ we are dealing with Coulomb scattering from a static point charge Ze

a) the transition amplitude is $T_{fi} = -i \int j^\mu_{fi} A_\mu \, d^4x$

j^μ_{fi} is the "transition current" and we can write it as

$j^\mu_{fi} = ie(\phi_f^* \partial^\mu \phi_i - (\partial^\mu \phi_f^*) \phi_i)$ where ϕ_f and ϕ_i are the plane-wave free particle solutions:

$$\phi_i = N_i e^{-ip_i x}$$
$$\phi_f = N_f e^{ip_f x}$$

Putting the explicit form of ϕ_f and ϕ_i in j^μ_{fi} we obtain

$$j^\mu_{fi} = eN_i N_f^* (p_f + p_i) e^{i(p_f - p_i)x}$$

$$\Rightarrow T_{fi} = -i \int eN_i N_f^* (p_f^\mu + p_i^\mu) A_\mu e^{i(p_f - p_i)x} d^4x$$

But $A_\mu = (V, \bar{0}) \Rightarrow p^\mu A_\mu = EV$

$$T_{fi} = -i \int eN_i N_f^* (E_f + E_i) V e^{i(p_f - p_i)x} d^4x$$

$$= -i \underbrace{\int dt\, e^{+i(E_f - E_i)t}}_{2\pi\delta(E_f - E_i)} \int d^3x\, eN_i N_f^* (E_f + E_i) V(x) e^{-i(\bar{p}_f - \bar{p}_i)\bar{x}} =$$

$\Downarrow$

Energy conservation

$$= -i N_i N_f^* (E_f + E_i) 2\pi \delta(E_f - E_i) \int \frac{Ze^2}{4\pi|\bar{x}|} e^{-i(\bar{p}_f - \bar{p}_i)\bar{x}} d^3x$$

the second spatial integral is the Fourier Transform of $\frac{1}{4\pi|x|}$

$$\frac{1}{|\bar{P}_f-\bar{P}_i|^2} = \int d^3x \, e^{i|\bar{P}_f-\bar{P}_i|\bar{x}} \frac{1}{4\pi|\bar{x}|}$$

$$T_{fi} = -iN_iN_f^*(E_f-E_i)2\pi\delta(E_f-E_i)\frac{Ze^2}{|\bar{P}_f-\bar{P}_i|^2} \quad ✓$$

b) The transition probability per time unit, i.e. the transition rate per time unit, is equal to $W_{fi} = \frac{|T_{fi}|^2}{T}$

where T is the time in which the interaction occurs.

But, the δ function expresses the conservation of energy between initial and final state, and from the Uncertainty Principle it can be inferred that the transition between 2 exact defined states with E_i and E_f must be infinitely separated in time

$$W_{fi} = \lim_{T\to\infty} \frac{|T_{fi}|^2}{T}$$

Assuming that the interaction occurs during a time period T from $t = -T/2$ up to $T = +T/2$

then, computing the squared module of T_{fi} we get the product of 2 δ functions, that we can express in form of integral

$$W_{fi} = \lim_{T\to\infty}\frac{1}{T}|V_{fi}|^2 \underbrace{\int_{-\infty}^{+\infty} dt\, e^{i(E_f-E_i)t}}_{2\pi\delta(E_f-E_i)} \int_{-T/2}^{T/2} dt'\, e^{i(E_f-E_i)t'}$$

$\downarrow$

this implies that we get a contribution in the second integral only if $E_f = E_i$

where $V_{fi} = -iN_iN_f^*(E_f-E_i)\frac{Ze^2}{|\bar{P}_f-\bar{P}_i|^2}$

$$W_{fi} = \lim_{T\to\infty} |V_{fi}|^2 2\pi\, \delta(E_f - E_i)\, \frac{1}{T}\int_{-T/2}^{T/2} dt'$$

$$= 2\pi\, |V_{fi}|^2\, \delta(E_f - E_i)$$

$$= |N_i N_f|^2\, 2\pi\, \delta(E_f - E_i) \left(\frac{2e^2 (E_f + E_i)}{|\bar{p}_f - \bar{p}_i|^2} \right)^2 \quad ✓$$

c) Compute the cross section

$$d\sigma = \frac{w_{fi}}{flux}\, dLips$$

the flux of incident particles is : $flux = |\bar{v}| \cdot \frac{2E_i}{V} = \frac{|\bar{p}_i|}{E_i} \frac{2E_i}{V} = 2\frac{|\bar{p}_i|}{V}$

being $\frac{2E_i}{V}$ the number of particle per volume unit and $\bar{v}$ the particle velocity

[N.B: We are adopting the "covariant" normalization of $2E$ particles per volume V]

the number of final states is : $dLips = \frac{V}{(2\pi)^3} \frac{d^3p_f}{2E_f}$ ($Lips$ = Lorentz invariant phase space)

$N = \frac{1}{\sqrt{V}}$ → it comes from $\int_V \rho\, d^3x = 2E$ where $\rho = 2E|N|^2$ (again using the "covariant" normalization)

the differential cross section $d\sigma$ is :

$$d\sigma = \frac{W_{fi}}{flux} \cdot dLips = \frac{1}{\cancel{V^2}}\, 2\pi\, \delta(E_f - E_i) \left(\frac{2e^2(E_f+E_i)}{|\bar{p}_f - \bar{p}_i|^2} \right)^2 \frac{\cancel{V}}{2|\bar{p}_i|} \cdot \frac{\cancel{V}}{(2\pi)^{\cancel{3}\,2}} \frac{d^3p_f}{2E_f}$$

From energy and momentum conservation

$E_f = E_i = E$, $|\bar{p}_f| = |\bar{p}_i| = p$

Also $d^3p_f = p_f^2\, dp_f\, d\Omega = p^2\, dp\, d\Omega$

$$d\sigma = \frac{1}{(2\pi)^2}\, \delta(E_f - E_i) \left(\frac{2e^2(E_f+E_i)}{|\bar{p}_f - \bar{p}_i|^2} \right)^2 \frac{p^2\, dp\, d\Omega}{2|\bar{p}_i|\, 2E_f}$$

$$= \frac{1}{(2\pi)^2}\, \delta(E_f - E_i) \left(\frac{2e^2(E_f+E_i)}{\underbrace{2p^2(1-\cos\theta)}_{4p^2 \operatorname{sen}^2\frac{\theta}{2}}} \right)^2 \frac{p\, dp\, d\Omega}{4E}$$

Since $E^2 = m^2 + p^2 \Rightarrow pdp = EdE$

and $\dfrac{pdpd\Omega}{4E}\,\delta(E_f - E_i) = \dfrac{dE}{4}\,d\Omega\,\delta(E_f - E_i) = \dfrac{d\Omega}{4}$

$$d\sigma = \left(\frac{2e^2 E}{4\pi |p|^2 \sin^2\theta_{/2} \cdot 2}\right)^2 d\Omega$$

or $\dfrac{d\sigma}{d\Omega} = \dfrac{z^2 e^4 E^2}{16\pi^2 |p|^4 \sin^4\theta_{/2} \cdot 4} = \dfrac{z^2 E^2 \alpha^2}{4|p|^4 \sin^4\theta_{/2}}$ where $\alpha = \dfrac{e^2}{4\pi} = \dfrac{1}{137}$

this is the Rutherford formula for relativistic kinematics

$\Rightarrow \sin^{-4}\theta_{/2}$ angular dependence ✓

d) the non-relativistic limit is obtained by considering $E = M$

and $E_{kin} = \dfrac{|P|^2}{2M}$

$$\frac{d\sigma}{d\Omega} = \frac{z^2 M^2 \alpha^2}{4M^2 E_{kin}^2 \sin^4\theta_{/2}} = \frac{z^2\alpha^2}{4E_{kin}^2 \sin^4\theta_{/2}}$$ ✓