Universität Zürich[UZH]

Physik-Institut

Kern- und Teilchenphysik II
Lecture 3: Weak Interaction

(adapted from the Handout of Prof. Mark Thomson)

Prof. Nico Serra

Dr. Marcin Chrzaszcz

Dr. Annapaola De Cosa (guest lecturer)

http://www.physik.uzh.ch/de/lehre/PHY211/HS2016.html

Boson Polarization

★ In this handout we are going to consider the decays of W and Z bosons, for this we will need to consider the polarization. Here simply quote results

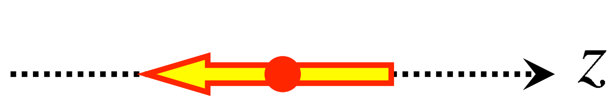
★ A real (i.e. not virtual) massless spin-1 boson can exist in two transverse polarization states, a massive spin-1 boson also can be longitudinally polarized

★ Boson wave-functions are written in terms of the polarization four-vector ε^μ

$$\boxed{B^\mu = \varepsilon^\mu e^{-ip.x} = \varepsilon^\mu e^{i(\vec{p}.\vec{x}-Et)}}$$

★ For a spin-1 boson travelling along the z-axis, the polarization four vectors are:

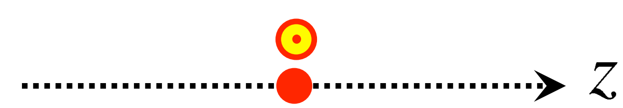
$$\varepsilon_-^\mu = \frac{1}{\sqrt{2}}(0,1,-i,0);$$



$$\boxed{S_z = -1}$$

transverse

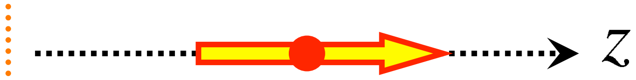
$$\varepsilon_L = \frac{1}{m}(p_z,0,0,E)$$



$$\boxed{S_z = 0}$$

longitudinal

$$\varepsilon_+^\mu = -\frac{1}{\sqrt{2}}(0,1,i,0)$$



$$\boxed{S_z = +1}$$

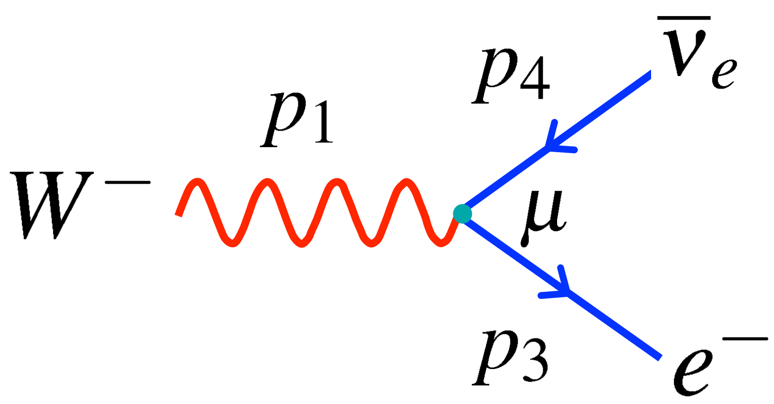
transverse

Longitudinal polarization isn't present for on-shell massless particles, the photon can exist in two helicity states $h = \pm 1$ (LH and RH circularly polarized light)

W-boson decay

★ **To calculate the W-Boson decay rate first consider** $W^- \to e^- \overline{\nu}_e$

★ **Want matrix element for :**



Incoming W-boson : $\varepsilon_\mu(p_1)$

Out-going electron : $\overline{u}(p_3)$

Out-going $\overline{\nu}_e$ **:** $v(p_4)$

Vertex factor $: -i\dfrac{g_W}{\sqrt{2}}\dfrac{1}{2}\gamma^\mu(1-\gamma^5)$

$$-iM_{fi} = \varepsilon_\mu(p_1).\overline{u}(p_3). -i\frac{g_W}{\sqrt{2}}\gamma^\mu \tfrac{1}{2}(1-\gamma^5).v(p_4)$$

Note, no propagator

$$\Rightarrow \quad \boxed{M_{fi} = \frac{g_W}{\sqrt{2}}\varepsilon_\mu(p_1)\overline{u}(p_3)\gamma^\mu \tfrac{1}{2}(1-\gamma^5)v(p_4)}$$

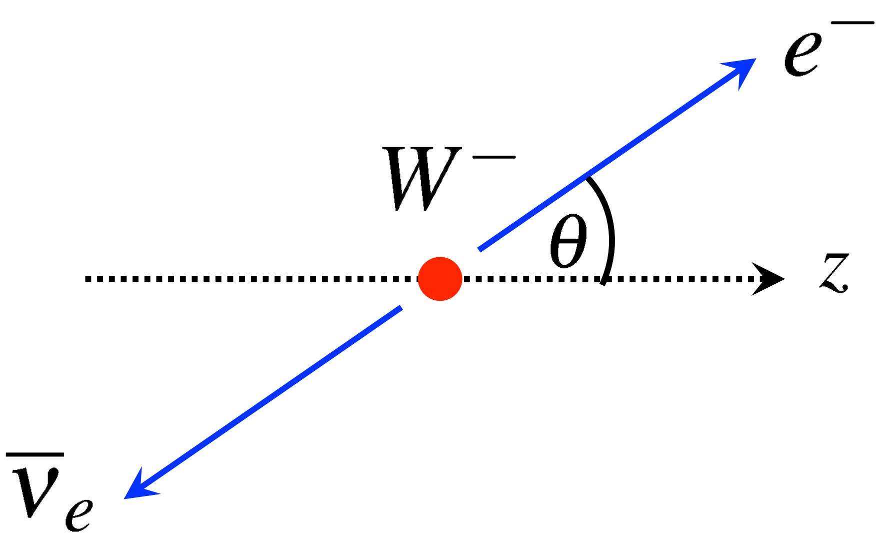
★ **This can be written in terms of the four-vector scalar product of the W-boson polarization** $\varepsilon_\mu(p_1)$ **and the weak charged current** j^μ

$$\boxed{M_{fi} = \frac{g_W}{\sqrt{2}}\varepsilon_\mu(p_1).j^\mu} \quad \textbf{with} \quad \boxed{j^\mu = \overline{u}(p_3)\gamma^\mu \tfrac{1}{2}(1-\gamma^5)v(p_4)}$$

W-decay

★ **First consider the lepton current** $j^\mu = \overline{u}(p_3)\gamma^\mu \frac{1}{2}(1-\gamma^5)v(p_4)$

★ **Work in Centre-of-Mass frame**



$$p_1 = (m_W, 0, 0, 0);$$
$$p_3 = (E, E\sin\theta, 0, E\cos\theta)$$
$$p_4 = (E, -E\sin\theta, 0, -E\cos\theta)$$

with $E = \dfrac{m_W}{2}$

★ **In the ultra-relativistic limit only LH particles and RH anti-particles participate in the weak interaction so**

$$j^\mu = \overline{u}(p_3)\gamma^\mu \tfrac{1}{2}(1-\gamma^5)v(p_4) = \overline{u}_\downarrow(p_3)\gamma^\mu v_\uparrow(p_4)$$

Note: $\frac{1}{2}(1-\gamma^5)v(p_4) = v_\uparrow(p_4)$ $\qquad$ $\overline{u}(p_3)\gamma^\mu v_\uparrow(p_4) = \overline{u}_\downarrow(p_3)\gamma^\mu v_\uparrow(p_4)$

Chiral projection operator, e.g. see p.131 or p.294

"Helicity conservation", e.g. see p.133 or p.295

W-decay

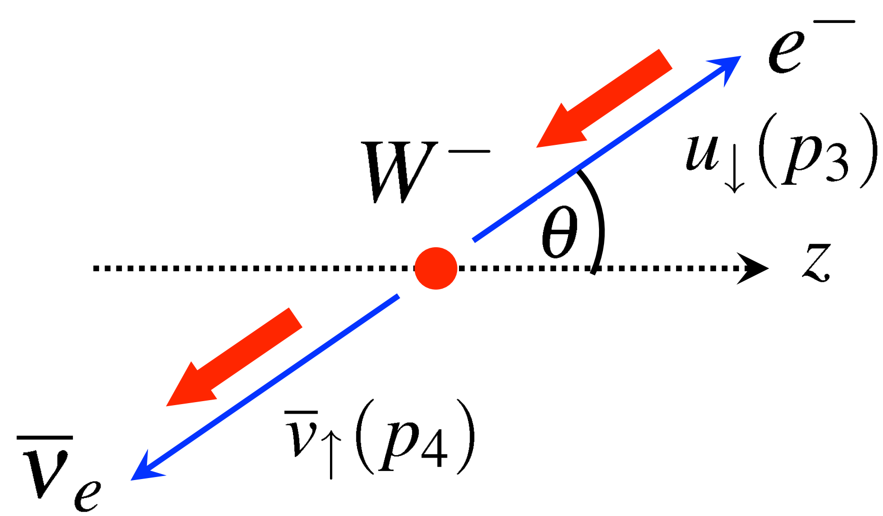
- **We have already calculated the current**

$$j^\mu = \overline{u}_\downarrow(p_3)\gamma^\mu v_\uparrow(p_4)$$

when considering $e^+e^- \to \mu^+\mu^-$

- **From page 128 we have for** $\mu_L^-\mu_R^+$

$$\boxed{j^\mu_{\uparrow\downarrow} = 2E(0, -\cos\theta, -i, \sin\theta)}$$

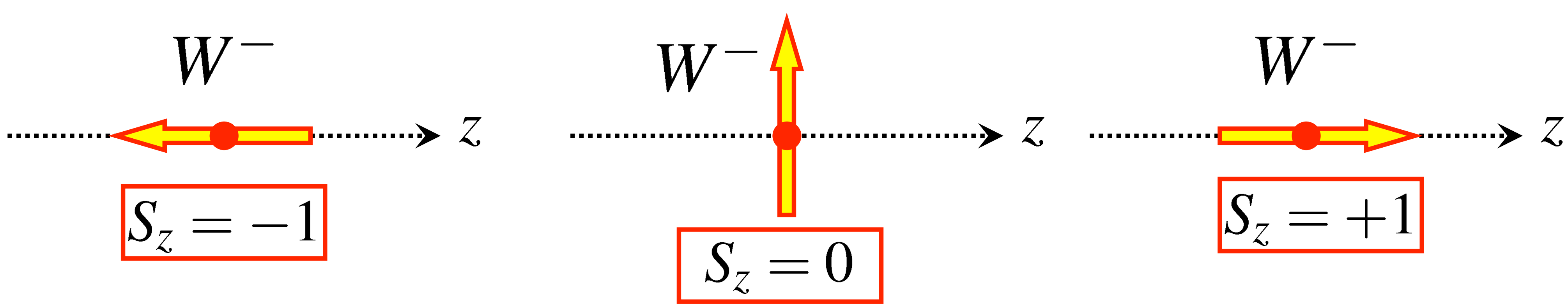


- **For the charged current weak Interaction we only have to consider this single combination of helicities**

$$j^\mu = \overline{u}(p_3)\gamma^\mu \tfrac{1}{2}(1-\gamma^5)v(p_4) = \overline{u}_\downarrow(p_3)\gamma^\mu v_\uparrow(p_4) = 2E(0, -\cos\theta, -i, \sin\theta)$$

and the three possible W-Boson polarization states:

$$\varepsilon^\mu_- = \frac{1}{\sqrt{2}}(0,1,-i,0); \quad \varepsilon_L = \frac{1}{m}(p_z,0,0,E) \quad \varepsilon^\mu_+ = -\frac{1}{\sqrt{2}}(0,1,i,0)$$



W-decay

★ **For a W-boson at rest these become:**

$$\epsilon_-^\mu = \frac{1}{\sqrt{2}}(0,1,-i,0); \quad \epsilon_L = (0,0,0,1) \quad \epsilon_+^\mu = -\frac{1}{\sqrt{2}}(0,1,i,0)$$

★ **Can now calculate the matrix element for the different polarization states**

$$M_{fi} = \frac{g_W}{\sqrt{2}}\epsilon_\mu(p_1)j^\mu \quad \textbf{with} \quad j^\mu = 2\frac{m_W}{2}(0,-\cos\theta,-i,\sin\theta)$$

Decay at rest : $E_e = E_\nu = m_W/2$

★ **giving**

$$\boxed{\epsilon_-} \quad M_- = \frac{g_W}{\sqrt{2}}\frac{1}{\sqrt{2}}(0,1,-i,0).m_W(0,-\cos\theta,-i,\sin\theta) = \frac{1}{2}g_W m_W(1+\cos\theta)$$

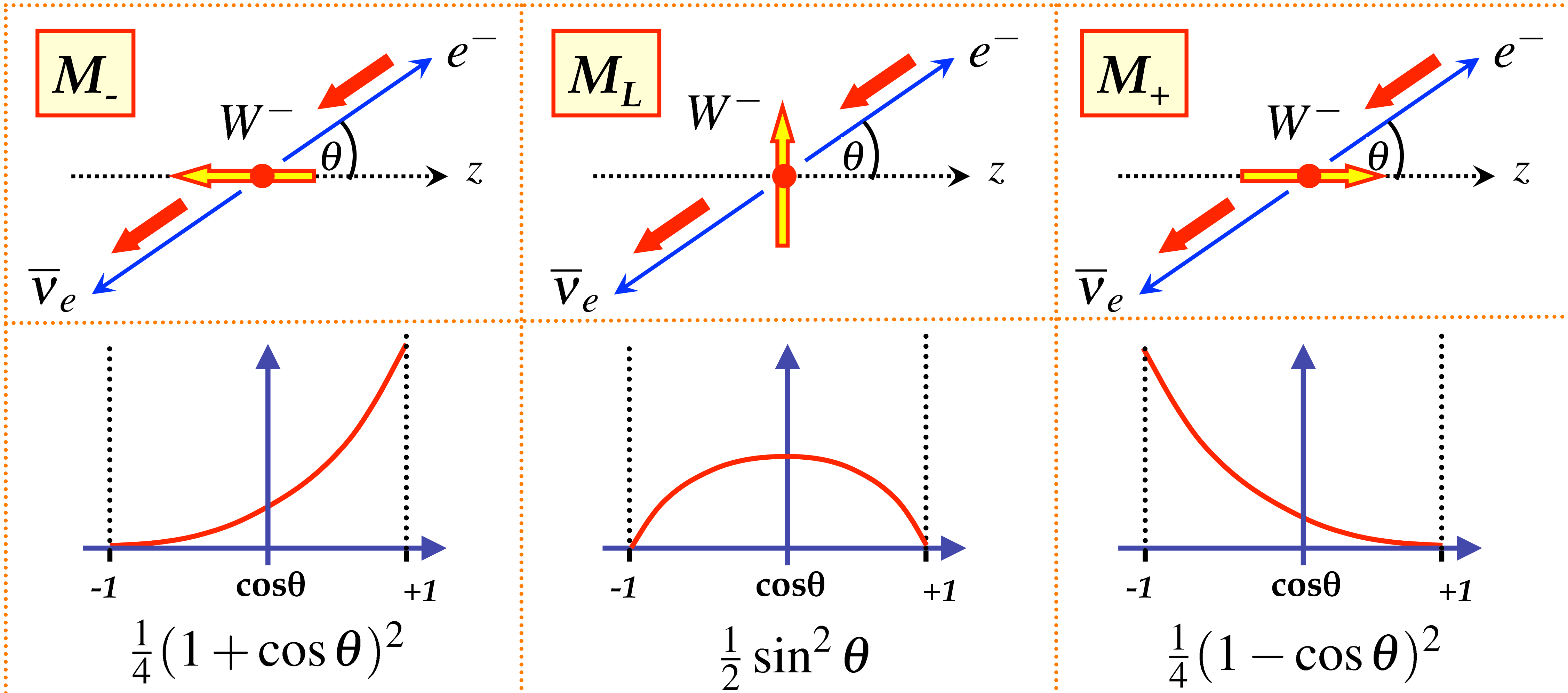
$$\boxed{\epsilon_L} \quad M_L = \frac{g_W}{\sqrt{2}}(0,0,0,1).m_W(0,-\cos\theta,-i,\sin\theta) = -\frac{1}{\sqrt{2}}g_W m_W \sin\theta$$

$$\boxed{\epsilon_+} \quad M_+ = -\frac{g_W}{\sqrt{2}}\frac{1}{\sqrt{2}}(0,1,i,0).m_W(0,-\cos\theta,-i,\sin\theta) = \frac{1}{2}g_W m_W(1-\cos\theta)$$

$$\Rightarrow \quad \boxed{\begin{aligned} |M_-|^2 &= g_W^2 m_W^2 \tfrac{1}{4}(1+\cos\theta)^2 \\ |M_L|^2 &= g_W^2 m_W^2 \tfrac{1}{2}\sin^2\theta \\ |M_+|^2 &= g_W^2 m_W^2 \tfrac{1}{4}(1-\cos\theta)^2 \end{aligned}}$$

W-decay

★ **The angular distributions can be understood in terms of the spin of the particles**



★ **The differential decay rate (see page 26) can be found using:**

$$\frac{d\Gamma}{d\Omega} = \frac{|p^*|}{32\pi^2 m_W^2}|M|^2$$

where p* is the C.o.M momentum of the final state particles, here $p^* = \frac{m_W}{2}$

W-decay

★ **Hence for the three different polarisations we obtain:**

$$\frac{d\Gamma_-}{d\Omega} = \frac{g_W^2 m_w}{64\pi^2}\frac{1}{4}(1+\cos\theta)^2 \qquad \frac{d\Gamma_L}{d\Omega} = \frac{g_W^2 m_w}{64\pi^2}\frac{1}{2}\sin^2\theta \qquad \frac{d\Gamma_+}{d\Omega} = \frac{g_W^2 m_w}{64\pi^2}\frac{1}{4}(1-\cos\theta)^2$$

★ **Integrating over all angles using**

$$\int \tfrac{1}{4}(1\pm\cos\theta)^2 d\phi\, d\cos\theta = \int \tfrac{1}{2}\sin^2\theta\, d\phi\, d\cos\theta = \frac{4\pi}{3}$$

★ **Gives**

$$\boxed{\Gamma_- = \Gamma_L = \Gamma_+ = \frac{g_W^2 m_W}{48\pi}}$$

★ **The total W-decay rate is independent of polarization; this has to be the case as the decay rate cannot depend on the arbitrary definition of the z-axis**

★ **For a sample of unpolarized W boson each polarization state is equally likely, for the average matrix element sum over all possible matrix elements and average over the three initial polarization states**

$$\begin{aligned}\langle |M_{fi}|^2 \rangle &= \tfrac{1}{3}(|M_-|^2 + |M_L|^2 + |M_+|^2) \\ &= \tfrac{1}{3} g_W^2 m_W^2 \left[\tfrac{1}{4}(1+\cos\theta)^2 + \tfrac{1}{2}\sin^2\theta + \tfrac{1}{4}(1-\cos\theta)^2\right] \\ &= \tfrac{1}{3} g_W^2 m_W^2\end{aligned}$$

★ **For a sample of unpolarized W-bosons, the decay is isotropic (as expected)**

W-decay

★ For this isotropic decay

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}\Omega} = \frac{|p^*|}{32\pi^2 m_W^2}\langle|M|^2\rangle \quad \Rightarrow \quad \Gamma = \frac{4\pi|p^*|}{32\pi^2 m_W^2}\langle|M|^2\rangle$$

$$\Rightarrow \quad \boxed{\Gamma(W^- \to e^-\overline{\nu}) = \frac{g_W^2 m_W}{48\pi}}$$

★ The calculation for the other decay modes (neglecting final state particle masses) is same. For quarks need to account for **colour** and **CKM matrix**. No decays to top – the top mass (175 GeV) is greater than the W-boson mass (80 GeV)

$$\begin{array}{llll}
W^- \to e^-\overline{\nu}_e & W^- \to d\overline{u} & \times 3|V_{ud}|^2 & W^- \to d\overline{c} & \times 3|V_{cd}|^2 \\
W^- \to \mu^-\overline{\nu}_\mu & W^- \to s\overline{u} & \times 3|V_{us}|^2 & W^- \to s\overline{c} & \times 3|V_{cs}|^2 \\
W^- \to \tau^-\overline{\nu}_\tau & W^- \to b\overline{u} & \times 3|V_{ub}|^2 & W^- \to b\overline{c} & \times 3|V_{cb}|^2
\end{array}$$

★ Unitarity of CKM matrix gives, e.g. $|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1$

★ Hence $BR(\mathrm{W} \to qq') = 6BR(\mathrm{W} \to \mathrm{e}\nu)$

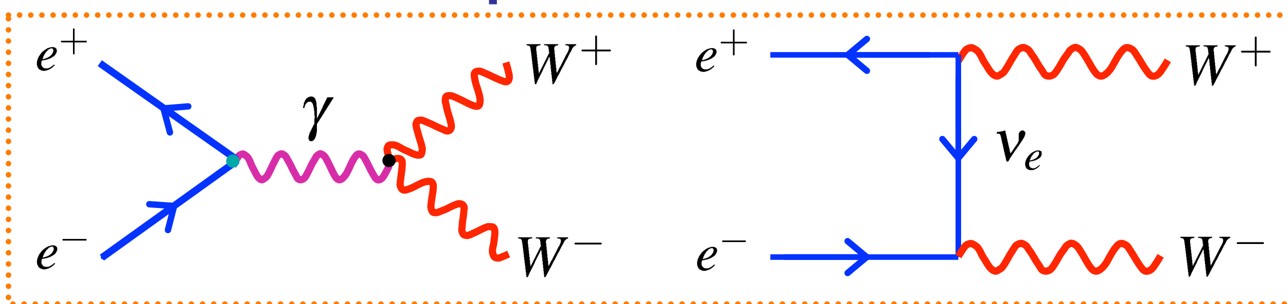
and thus the total decay rate :

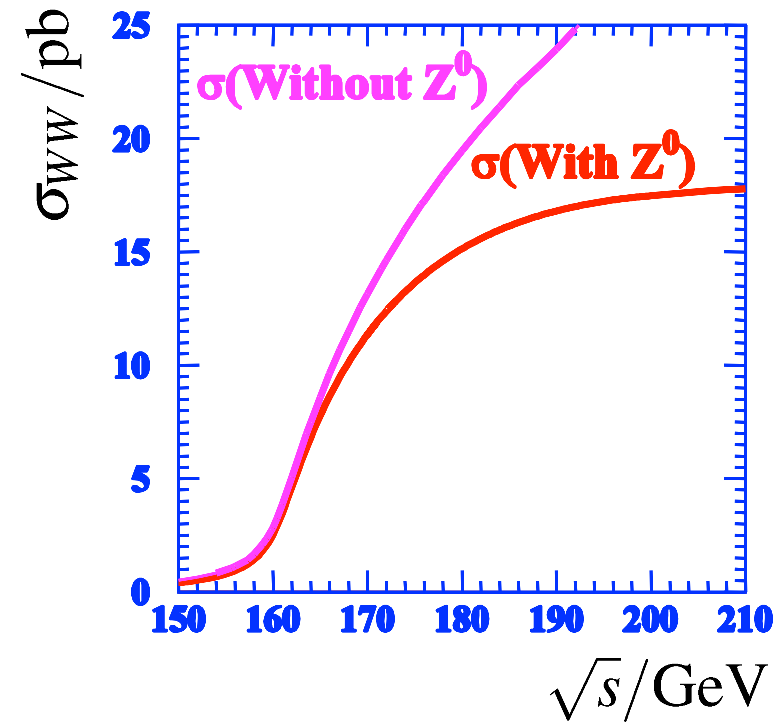
$$\boxed{\Gamma_W = 9\Gamma_{W\to e\nu} = \frac{3g_W^2 m_W}{16\pi} = 2.07\,\mathrm{GeV}}$$

Experiment: **2.14±0.04 GeV**
(our calculation neglected a 3% QCD correction to decays to quarks)

From W to Z

- ★ The $W^{\pm}$ bosons carry the EM charge - suggestive Weak are EM forces are related.
- ★ W bosons can be produced in e^+e^- annihilation

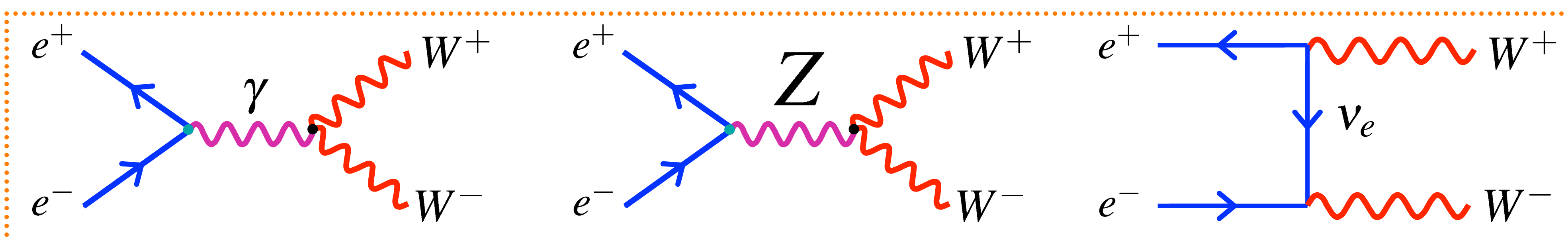




- ★ With just these two diagrams there is a problem: the cross section increases with C.o.M energy and at some point violates **QM unitarity**

UNITARITY VIOLATION: when QM calculation gives larger flux of W bosons than incoming flux of electrons/positrons

- ★ Problem can be "fixed" by introducing a new boson, the Z. The new diagram interferes negatively with the above two diagrams fixing the unitarity problem



$$|M_{\gamma WW} + M_{ZWW} + M_{\nu WW}|^2 < |M_{\gamma WW} + M_{\nu WW}|^2$$

- ★ Only works if **Z, γ, W** couplings are related: need **ELECTROWEAK UNIFICATION**

Reminder

Symmetries and Conservation Laws

★ **Suppose physics is invariant under the transformation**

$$\psi \rightarrow \psi' = \hat{U}\psi$$ **e.g. rotation of the coordinate axes**

• **To conserve probability normalisation require**

$$\langle\psi|\psi\rangle = \langle\psi'|\psi'\rangle = \langle\hat{U}\psi|\hat{U}\psi\rangle = \langle\psi|\hat{U}^\dagger\hat{U}|\psi\rangle$$

$$\rightarrow \boxed{\hat{U}^\dagger\hat{U} = 1}$$ **i.e.** $\hat{U}$ **has to be unitary**

• **For physical predictions to be unchanged by the symmetry transformation, also require all QM matrix elements unchanged**

$$\boxed{\langle\psi|\hat{H}|\psi\rangle = \langle\psi'|\hat{H}|\psi'\rangle} = \langle\psi|\hat{U}^\dagger\hat{H}\hat{U}|\psi\rangle$$

i.e. require $\hat{U}^\dagger\hat{H}\hat{U} = \hat{H}$

$\times\hat{U}$ $\hat{U}\hat{U}^\dagger\hat{H}\hat{U} = \hat{U}\hat{H} \rightarrow \hat{H}\hat{U} = \hat{U}\hat{H}$

therefore $\boxed{[\hat{H},\hat{U}] = 0}$ $\boxed{\hat{U} \textbf{ commutes with the Hamiltonian}}$

★ **Now consider the infinitesimal transformation** **(** ε **small)**

$$\hat{U} = 1 + i\varepsilon\hat{G}$$

($\hat{G}$ **is called the generator of the transformation)**

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Symmetries and Conservation Laws

- For $\hat{U}$ to be unitary

$$\hat{U}\hat{U}^\dagger = (1+i\varepsilon\hat{G})(1-i\varepsilon\hat{G}^\dagger) = 1 + i\varepsilon(\hat{G}-\hat{G}^\dagger) + O(\varepsilon^2)$$

neglecting terms in ε^2 $\quad UU^\dagger = 1 \quad \rightarrow \quad \boxed{\hat{G} = \hat{G}^\dagger}$

i.e. $\hat{G}$ is Hermitian and therefore corresponds to an observable quantity G !

- Furthermore, $[\hat{H},\hat{U}] = 0 \quad \Rightarrow [\hat{H}, 1+i\varepsilon\,\hat{G}] = 0 \quad \Rightarrow [\hat{H},\hat{G}] = 0$

But from QM $\quad \frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{G}\rangle = i\langle[\hat{H},\hat{G}]\rangle = 0$

i.e. G is a conserved quantity.

Symmetry ⟷ Conservation Law

★ **For each symmetry of nature have an observable <u>conserved</u> quantity**

<u>Example:</u> Infinitesimal spatial translation $x \rightarrow x+\varepsilon$

i.e. expect physics to be invariant under $\psi(x) \rightarrow \psi' = \psi(x+\varepsilon)$

$$\psi'(x) = \psi(x+\varepsilon) = \psi(x) + \frac{\partial\psi}{\partial x}\varepsilon = \left(1+\varepsilon\frac{\partial}{\partial x}\right)\psi(x)$$

but $\quad \hat{p}_x = -i\frac{\partial}{\partial x} \quad \rightarrow \quad \psi'(x) = (1+i\varepsilon\hat{p}_x)\psi(x)$

The generator of the symmetry transformation is $\hat{p}_x$ $\rightarrow$ p_x is conserved

- **Translational invariance** of physics implies **momentum conservation !**

Symmetries and Conservation Laws

- In general the symmetry operation may depend on more than one parameter

$$\hat{U} = 1 + i\vec{\varepsilon}.\vec{G}$$

For example for an infinitesimal 3D linear translation : $\vec{r} \rightarrow \vec{r} + \vec{\varepsilon}$

$$\rightarrow \quad \hat{U} = 1 + i\vec{\varepsilon}.\vec{p} \qquad \vec{p} = (\hat{p}_x, \hat{p}_y, \hat{p}_z)$$

- So far have only considered an infinitesimal transformation, however a finite transformation can be expressed as a series of infinitesimal transformations

$$\hat{U}(\vec{\alpha}) = \lim_{n\rightarrow\infty}\left(1 + i\frac{\vec{\alpha}}{n}.\vec{G}\right)^n = e^{i\vec{\alpha}.\vec{G}}$$

<u>**Example:**</u> Finite spatial translation in 1D: $x \rightarrow x + x_0$ with $\hat{U}(x_0) = e^{ix_0\hat{p}_x}$

$$\psi'(x) = \psi(x + x_0) = \hat{U}\psi(x) = \exp\left(x_0\frac{\mathrm{d}}{\mathrm{d}x}\right)\psi(x) \qquad \left(p_x = -i\frac{\partial}{\partial x}\right)$$

$$= \left(1 + x_0\frac{\mathrm{d}}{\mathrm{d}x} + \frac{x_0^2}{2!}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \ldots\right)\psi(x)$$

$$= \psi(x) + x_0\frac{\mathrm{d}\psi}{\mathrm{d}x} + \frac{x_0^2}{2}\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + \ldots$$

i.e. obtain the expected Taylor expansion

Isospin

- The proton and neutron have very similar masses and the nuclear force is found to be approximately charge-independent, i.e.

$$V_{pp} \approx V_{np} \approx V_{nn}$$

- To reflect this symmetry, Heisenberg (1932) proposed that if you could "switch off" the electric charge of the proton

There would be no way to distinguish between a proton and neutron

- Proposed that the neutron and proton should be considered as two states of a single entity; the **nucleon**

$$p = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad n = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

★ Analogous to the spin-up/spin-down states of a spin-½ particle

ISOSPIN

★ Expect physics to be invariant under rotations in this space

- The neutron and proton form an isospin doublet with total isospin $I = ½$ and third component $I_3 = \pm ½$

Flavour Symmetry

We can extend this idea to the quarks:

★ **Assume the strong interaction treats all quark flavours equally (it does)**

- **Because $m_u \approx m_d$:**

  The strong interaction possesses an approximate flavour symmetry
  i.e. from the point of view of the strong interaction nothing changes if all up quarks are replaced by down quarks and *vice versa*.

- **Choose the basis** $$u = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad d = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- **Express the invariance of the strong interaction under $u \leftrightarrow d$ as invariance under "rotations" in an abstract isospin space**

$$\begin{pmatrix} u' \\ d' \end{pmatrix} = \hat{U} \begin{pmatrix} u \\ d \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix}$$

The 2x2 unitary matrix depends on 4 complex numbers, i.e. 8 real parameters
But there are four constraints from $\hat{U}^\dagger \hat{U} = 1$

➡ **8 – 4 = 4 independent matrices**

- **In the language of group theory the four matrices form the U(2) group**

Flavour Symmetry

- One of the matrices corresponds to multiplying by a phase factor

$$\hat{U}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} e^{i\phi}$$

not a flavour transformation and of no relevance here.

- The remaining three matrices form an SU(2) group (special unitary) with $\boxed{\det U = 1}$
- For an infinitesimal transformation, in terms of the Hermitian generators $\hat{G}$

$$\hat{U} = 1 + i\varepsilon\hat{G}$$

- $\det U = 1 \quad \Rightarrow \quad Tr(\hat{G}) = 0$
- A linearly independent choice for $\hat{G}$ are the Pauli spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- The proposed flavour symmetry of the strong interaction has the same transformation properties as SPIN !
- Define ISOSPIN: $\vec{T} = \frac{1}{2}\vec{\sigma} \qquad \hat{U} = e^{i\vec{\alpha}.\vec{T}}$
- Check this works, for an infinitesimal transformation

$$\hat{U} = 1 + \frac{1}{2}i\vec{\varepsilon}.\vec{\sigma} \quad = \quad 1 + \frac{i}{2}(\varepsilon_1\sigma_1 + \varepsilon_2\sigma_2 + \varepsilon_3\sigma_3) = \begin{pmatrix} 1 + \frac{1}{2}i\varepsilon_3 & \frac{1}{2}i(\varepsilon_1 - i\varepsilon_2) \\ \frac{1}{2}i(\varepsilon_1 + i\varepsilon_2) & 1 - \frac{1}{2}i\varepsilon_3 \end{pmatrix}$$

Which is, as required, unitary and has unit determinant

$$U^\dagger U = I + O(\varepsilon^2) \qquad \det U = 1 + O(\varepsilon^2)$$

Properties of Isospin

- Isospin has the exactly the same properties as spin

$$[T_1, T_2] = iT_3 \quad [T_2, T_3] = iT_1 \quad [T_3, T_1] = iT_2$$

$$[T^2, T_3] = 0 \qquad T^2 = T_1^2 + T_2^2 + T_3^2$$

As in the case of spin, have three non-commuting operators, T_1, T_2, T_3 and even though all three correspond to observables, can't know them simultaneously. So label states in terms of **total isospin** I and the third component of isospin I_3

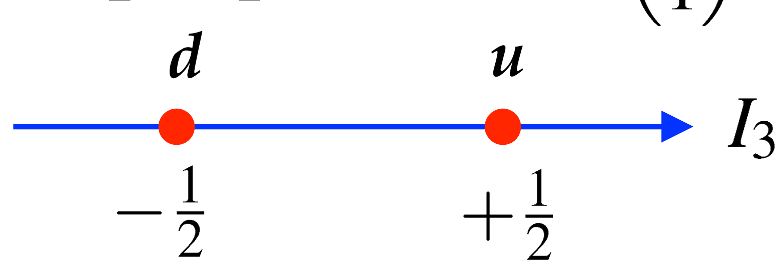
NOTE: isospin has nothing to do with spin – just the same mathematics

- The eigenstates are exact analogues of the eigenstates of ordinary angular momentum $|s, m\rangle \rightarrow |I, I_3\rangle$

with $$T^2|I, I_3\rangle = I(I+1)|I, I_3\rangle \qquad T_3|I, I_3\rangle = I_3|I, I_3\rangle$$

- In terms of isospin:

$$u = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |\ \tfrac{1}{2}, +\tfrac{1}{2}\ \rangle \qquad d = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |\tfrac{1}{2}, -\tfrac{1}{2}\rangle$$



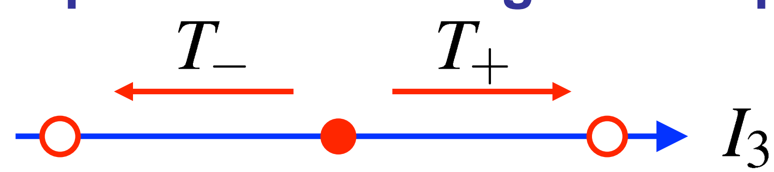
$$\boxed{I = \tfrac{1}{2}, \quad I_3 = \pm\tfrac{1}{2}}$$

- In general $I_3 = \frac{1}{2}(N_u - N_d)$

Properties of Isospin

- **Can define isospin ladder operators – analogous to spin ladder operators**

$\boxed{T_- \equiv T_1 - iT_2}$ **u → d**



$\boxed{T_+ \equiv T_1 + iT_2}$ **d → u**

$$T_+|I,I_3\rangle = \sqrt{I(I+1) - I_3(I_3+1)}|I,I_3+1\rangle$$
$$T_-|I,I_3\rangle = \sqrt{I(I+1) - I_3(I_3-1)}|I,I_3-1\rangle$$

Step up/down in I_3 until reach end of multiplet $T_+|I,+I\rangle = 0 \quad T_-|I,-I\rangle = 0$

$$\boxed{T_+u = 0 \qquad T_+d = u \qquad T_-u = d \qquad T_-d = 0}$$

- **Ladder operators turn $u \to d$ and $d \to u$**

★ **Combination of isospin: e.g. what is the isospin of a system of two d quarks, is exactly analogous to combination of spin (i.e. angular momentum)**

$$|I^{(1)}, I_3^{(1)}\rangle|I^{(2)}, I_3^{(2)}\rangle \to |I, I_3\rangle$$

- **I_3 additive :** $I_3 = I_3^{(1)} + I_3^{(2)}$
- **I in integer steps from** $|I^{(1)} - I^{(2)}|$ **to** $|I^{(1)} + I^{(2)}|$

★ **Assumed symmetry of Strong Interaction under isospin transformations implies the existence of conserved quantites**

- **In strong interactions I_3 and I are conserved, analogous to conservation of J_z and J for angular momentum**

$SU(2)_L$: Weak Interaction

★ The Weak Interaction arises from **SU(2)** local phase transformations

$$\psi \to \psi' = \psi e^{i\vec{\alpha}(x).\frac{\vec{\sigma}}{2}}$$

where the $\vec{\sigma}$ are the generators of the SU(2) symmetry, i.e the three Pauli spin matrices

→ $\boxed{\text{3 Gauge Bosons}}$ $W_1^\mu,\ W_2^\mu,\ W_3^\mu$

★ The wave-functions have two components which, in analogy with isospin, are represented by "weak isospin"

★ The fermions are placed in isospin doublets and the local phase transformation corresponds to

$$\begin{pmatrix}\nu_e\\ e^-\end{pmatrix} \to \begin{pmatrix}\nu_e\\ e^-\end{pmatrix}' = e^{i\vec{\alpha}(x).\frac{\vec{\sigma}}{2}}\begin{pmatrix}\nu_e\\ e^-\end{pmatrix}$$

★ Weak Interaction only couples to LH particles/RH anti-particles, hence only place LH particles/RH anti-particles in weak isospin doublets: $I_W = \frac{1}{2}$
RH particles/LH anti-particles placed in weak isospin singlets: $I_W = 0$

Weak Isospin

$$\boxed{I_W = \tfrac{1}{2}}\quad \begin{pmatrix}\nu_e\\ e^-\end{pmatrix}_L, \begin{pmatrix}\nu_\mu\\ \mu^-\end{pmatrix}_L, \begin{pmatrix}\nu_\tau\\ \tau^-\end{pmatrix}_L, \begin{pmatrix}u\\ d'\end{pmatrix}_L, \begin{pmatrix}c\\ s'\end{pmatrix}_L, \begin{pmatrix}t\\ b'\end{pmatrix}_L \quad \begin{matrix}\leftarrow \boxed{I_W^3 = +\frac{1}{2}}\\ \leftarrow \boxed{I_W^3 = -\frac{1}{2}}\end{matrix}$$

$$\boxed{I_W = 0}\quad (\nu_e)_R,\ (e^-)_R, \ldots (u)_R,\ (d)_R, \ldots$$

Note: RH/LH refer to chiral states

SU(2)$_L$: Weak Interaction

★ **For simplicity only consider** $\chi_L = \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L$

• **The gauge symmetry specifies the form of the interaction: one term for each of the 3 generators of SU(2) – [note: here include interaction strength in current]**

$$j^1_\mu = g_W \overline{\chi}_L \gamma^\mu \tfrac{1}{2}\sigma_1 \chi_L \qquad j^2_\mu = g_W \overline{\chi}_L \gamma^\mu \tfrac{1}{2}\sigma_2 \chi_L \qquad j^3_\mu = g_W \overline{\chi}_L \gamma^\mu \tfrac{1}{2}\sigma_3 \chi_L$$

★ **The charged current W$^+$/W$^-$ interaction enters as a linear combinations of W$_1$, W$_2$**

$$W^{\pm\mu} = \frac{1}{\sqrt{2}}(W_1^\mu \pm W_2^\mu)$$

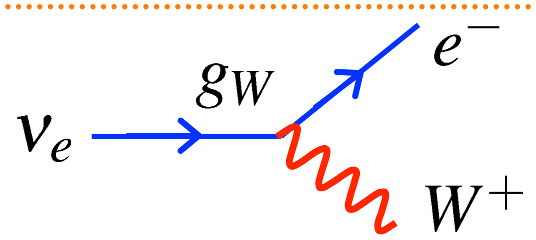
★ **The W$^\pm$ interaction terms**

$$j^\mu_\pm = \frac{g_W}{\sqrt{2}}(j_1^\mu \pm i j_2^\mu) = \frac{g_W}{\sqrt{2}} \overline{\chi}_L \gamma^\mu \tfrac{1}{2}(\sigma_1 \pm i\sigma_2)\chi_L$$

★ **Express in terms of the weak isospin ladder operators** $\sigma_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$

$$\left.\boxed{j^\mu_\pm = \frac{g_W}{\sqrt{2}} \overline{\chi}_L \gamma^\mu \sigma_\pm \chi_L}\ \right\}$$ **Origin of $\frac{1}{\sqrt{2}}$ in Weak CC**

W$^+$



corresponds to $\boxed{j^\mu_+ = \frac{g_W}{\sqrt{2}} \overline{\chi}_L \gamma^\mu \sigma_+ \chi_L}$

Bars indicates adjoint spinors

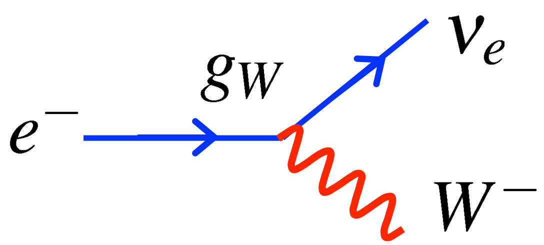
which can be understood in terms of the weak isospin doublet

$$j^\mu_+ = \frac{g_W}{\sqrt{2}} \overline{\chi}_L \gamma^\mu \sigma_+ \chi_L = \frac{g_W}{\sqrt{2}} (\overline{\nu}_L, \overline{e}_L)\gamma^\mu \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \nu \\ e \end{pmatrix}_L = \frac{g_W}{\sqrt{2}} \overline{\nu}_L \gamma^\mu e_L = \boxed{\frac{g_W}{\sqrt{2}} \overline{\nu} \gamma^\mu \tfrac{1}{2}(1-\gamma^5) e}$$

SU(2)$_L$: Weak Interaction

★ **Similarly**

W⁻



corresponds to

$$\boxed{j_-^\mu = \frac{g_W}{\sqrt{2}}\overline{\chi}_L\gamma^\mu\sigma_-\chi_L}$$

$$j_-^\mu = \frac{g_W}{\sqrt{2}}\overline{\chi}_L\gamma^\mu\sigma_-\chi_L = \frac{g_W}{\sqrt{2}}(\overline{\nu}_L, \overline{e}_L)\gamma^\mu \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \nu \\ e \end{pmatrix}_L = \frac{g_W}{\sqrt{2}}\overline{e}_L\gamma^\mu\nu_L = \frac{g_W}{\sqrt{2}}\overline{e}\gamma^\mu\tfrac{1}{2}(1-\gamma^5)\nu$$

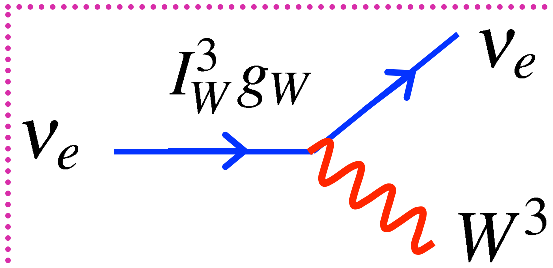
★ **However have an additional interaction due to W³**

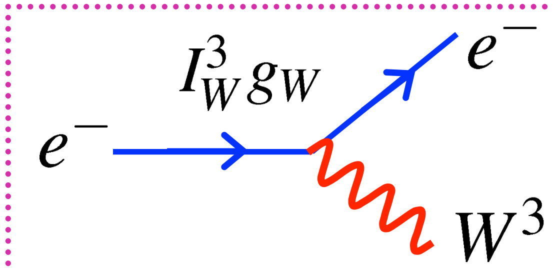
$$\boxed{j_3^\mu = g_W\overline{\chi}_L\gamma^\mu\tfrac{1}{2}\sigma_3\chi_L}$$

expanding this:

$$\boxed{I_W^3 = \pm\tfrac{1}{2}}$$

$$j_3^\mu = g_W\frac{1}{2}(\overline{\nu}_L, \overline{e}_L)\gamma^\mu \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \nu \\ e \end{pmatrix}_L = g_W\tfrac{1}{2}\overline{\nu}_L\gamma^\mu\nu_L - g_W\tfrac{1}{2}\overline{e}_L\gamma^\mu e_L$$





⇒ **NEUTRAL CURRENT INTERACTIONS !**

Electroweak Unification

- ★ Tempting to identify the W^3 as the Z
- ★ However this is not the case, have two physical neutral spin-1 gauge bosons, γ, Z and the W^3 is a mixture of the two,
- ★ Equivalently write the photon and Z in terms of the W^3 and a new neutral spin-1 boson the B
- ★ The physical bosons (the Z and photon field, A) are:

$$A_\mu = B_\mu \cos\theta_W + W^3_\mu \sin\theta_W$$

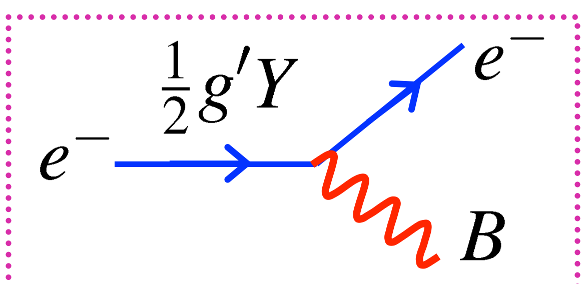
$$Z_\mu = -B_\mu \sin\theta_W + W^3_\mu \cos\theta_W$$

θ_W is the weak mixing angle

- ★ The new boson is associated with a new gauge symmetry similar to that of electromagnetism : $U(1)_Y$
- ★ The charge of this symmetry is called **WEAK HYPERCHARGE** Y

$$\boxed{Y = 2Q - 2I^3_W}$$

Q is the EM charge of a particle
I^3_W is the third comp. of weak isospin



- By convention the coupling to the B_μ is $\frac{1}{2}g'Y$

$e_L: \; Y = 2(-1) - 2(-\tfrac{1}{2}) = -1$ $\qquad$ $\nu_L: \; Y = +1$

$e_R: \; Y = 2(-1) - 2(0) = -2$ $\qquad$ $\nu_R: \; Y = 0$

(this identification of hypercharge in terms of Q and I_3 makes all of the following work out)

Electroweak Unification

★ **For this to work the coupling constants of the W³, B, and photon must be related**

e.g. consider contributions involving the neutral interactions of electrons:

γ $$j_\mu^{em} = e\overline{\psi}Q_e\gamma_\mu\psi = e\overline{\mathrm{e}}_L Q_\mathrm{e}\gamma_\mu \mathrm{e}_L + e\overline{\mathrm{e}}_R Q_e\gamma_\mu \mathrm{e}_R$$

W³ $$j_\mu^{W^3} = -\frac{g_W}{2}\overline{\mathrm{e}}_L\gamma_\mu \mathrm{e}_L$$

B $$j_\mu^{Y} = \frac{g'}{2}\overline{\psi}Y_e\gamma_\mu\psi = \frac{g'}{2}\overline{\mathrm{e}}_L Y_{\mathrm{e}_L}\gamma_\mu \mathrm{e}_L + \frac{g'}{2}\overline{\mathrm{e}}_R Y_{\mathrm{e}_R}\gamma_\mu \mathrm{e}_R$$

★ **The relation** $A_\mu = B_\mu\cos\theta_W + W_\mu^3\sin\theta_W$ **is equivalent to requiring**

$$\boxed{j_\mu^{em} = j_\mu^Y\cos\theta_W + j_\mu^{W^3}\sin\theta_W}$$

•**Writing this in full:**

$$e\overline{\mathrm{e}}_L Q_\mathrm{e}\gamma_\mu \mathrm{e}_L + e\overline{\mathrm{e}}_R Q_e\gamma_\mu \mathrm{e}_R = \tfrac{1}{2}g'\cos\theta_W[\overline{\mathrm{e}}_L Y_{\mathrm{e}_L}\gamma_\mu \mathrm{e}_L + \overline{\mathrm{e}}_R Y_{\mathrm{e}_R}\gamma_\mu \mathrm{e}_R] - \tfrac{1}{2}g_W\sin\theta_W[\overline{\mathrm{e}}_L\gamma_\mu e_L]$$

$$-e\overline{\mathrm{e}}_L\gamma_\mu \mathrm{e}_L - e\overline{\mathrm{e}}_R\gamma_\mu \mathrm{e}_R = \tfrac{1}{2}g'\cos\theta_W[-\overline{\mathrm{e}}_L\gamma_\mu \mathrm{e}_L - 2\overline{\mathrm{e}}_R\gamma_\mu \mathrm{e}_R] - \tfrac{1}{2}g_W\sin\theta_W[\overline{\mathrm{e}}_L\gamma_\mu e_L]$$

which works if: $$\boxed{e = g_W\sin\theta_W = g'\cos\theta_W}$$ **(i.e. equate coefficients of L and R terms)**

★ **Couplings of electromagnetism, the weak interaction and the interaction of the $U(1)_Y$ symmetry are therefore related.**

The Z-boson

★ **In this model we can now derive the couplings of the Z Boson**

$$Z_\mu = -B_\mu \sin\theta_W + W^3_\mu \cos\theta_W$$

$\boxed{I^3_W}$ **for the electron** $I^3_W = \frac{1}{2}$

$$j^Z_\mu = -\tfrac{1}{2} g' \sin\theta_W [\bar{e}_L Y_{e_L} \gamma_\mu e_L + \bar{e}_R Y_{e_R} \gamma_\mu e_R] - \tfrac{1}{2} g_W \cos\theta_W [e_L \gamma_\mu e_L]$$

•**Writing this in terms of weak isospin and charge:**

$$j^Z_\mu = -\tfrac{1}{2} g' \sin\theta_W [\bar{e}_L (2Q - 2I^3_W) \gamma_\mu e_L + \bar{e}_R \underbrace{(2Q)}_{\text{For RH chiral states } I_3=0} \gamma_\mu e_R] + I^3_W g_W \cos\theta_W [e_L \gamma_\mu e_L]$$

•**Gathering up the terms for LH and RH chiral states:**

$$j^Z_\mu = [g' I^3_W \sin\theta_W - g' Q \sin\theta_W + g_W I^3_W \cos\theta_W]\, \bar{e}_L \gamma_\mu e_L - [g' Q \sin\theta_W]\, e_R \gamma_\mu e_R$$

•**Using:** $e = g_W \sin\theta_W = g' \cos\theta_W$ **gives**

$$j^Z_\mu = \left[g' \frac{(I^3_W - Q\sin^2\theta_W)}{\sin\theta_W} \right] \bar{e}_L \gamma_\mu e_L - \left[g' \frac{Q \sin^2\theta_W}{\sin\theta_W} \right] e_R \gamma_\mu e_R$$

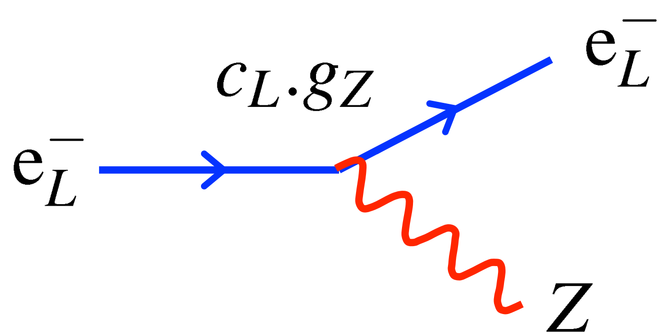
$$\boxed{j^Z_\mu = g_Z (I^3_W - Q\sin^2\theta_W)[\bar{e}_L \gamma_\mu e_L] - g_Z Q \sin^2\theta_W [e_R \gamma_\mu e_R]}$$

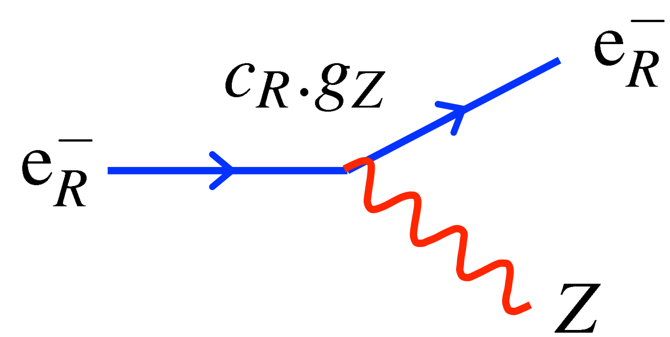
with $\boxed{e = g_Z \cos\theta_W \sin\theta_W}$ **i.e.** $\boxed{g_Z = \dfrac{g_W}{\cos\theta_W}}$

The Z-boson

★ **Unlike for the Charged Current Weak interaction (W) the Z Boson couples to both LH and RH chiral components, but not equally…**

$$
\begin{aligned}
j_\mu^Z &= g_Z(I_W^3 - Q\sin^2\theta_W)[\bar{\mathrm{e}}_L\gamma_\mu \mathrm{e}_L] - g_Z Q\sin^2\theta_W[\mathrm{e}_R\gamma_\mu e_R] \\
&= g_Z c_L[\bar{\mathrm{e}}_L\gamma_\mu \mathrm{e}_L] + g_Z c_R[\mathrm{e}_R\gamma_\mu e_R]
\end{aligned}
$$





$$\boxed{c_L = I_W^3 - Q\sin^2\theta_W} \qquad \boxed{c_R = -Q\sin^2\theta_W}$$

W³ part of Z couples only to LH components (like W±)

B_μ part of Z couples equally to LH and RH components

★ **Use projection operators to obtain vector and axial vector couplings**

$$\bar{u}_L\gamma_\mu u_L = \bar{u}\gamma_\mu \tfrac{1}{2}(1-\gamma_5)u \qquad \bar{u}_R\gamma_\mu u_R = \bar{u}\gamma_\mu \tfrac{1}{2}(1+\gamma_5)u$$

$$j_\mu^Z = g_Z \bar{u}\gamma_\mu\left[c_L\tfrac{1}{2}(1-\gamma_5) + c_R\tfrac{1}{2}(1+\gamma_5))\right]u$$

The Z-boson

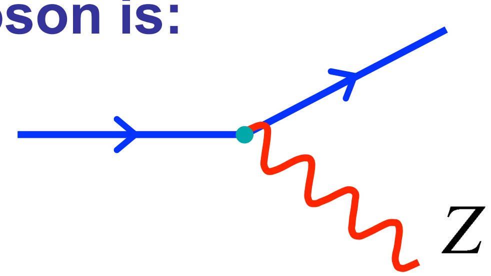
$$j_\mu^Z = \frac{g_Z}{2}\overline{u}\gamma_\mu\left[(c_L + c_R) + (c_R - c_L)\gamma_5)\right]u$$

★ **Which in terms of V and A components gives:** $$j_\mu^Z = \frac{g_Z}{2}\overline{u}\gamma_\mu\left[c_V - c_A\gamma_5\right]u$$

with $\boxed{c_V = c_L + c_R = I_W^3 - 2Q\sin^2\theta_W}$ $\boxed{c_A = c_L - c_R = I_W^3}$

★ **Hence the vertex factor for the Z boson is:**

$$\boxed{-ig_Z\tfrac{1}{2}\gamma_\mu\left[c_V - c_A\gamma_5\right]}$$



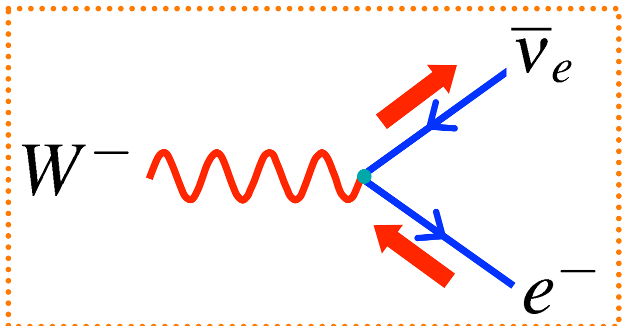
★ **Using the experimentally determined value of the weak mixing angle:**

$\boxed{\sin^2\theta_W \approx 0.23}$

Fermion	Q	I_W^3	c_L	c_R	c_V	c_A
ν_e, ν_μ, ν_τ	0	$+\frac{1}{2}$	$+\frac{1}{2}$	0	$+\frac{1}{2}$	$+\frac{1}{2}$
e^-, μ^-, τ^-	-1	$-\frac{1}{2}$	-0.27	0.23	-0.04	$-\frac{1}{2}$
u, c, t	$+\frac{2}{3}$	$+\frac{1}{2}$	0.35	-0.15	$+0.19$	$+\frac{1}{2}$
d, s, b	$-\frac{1}{3}$	$-\frac{1}{2}$	-0.42	0.08	-0.35	$-\frac{1}{2}$

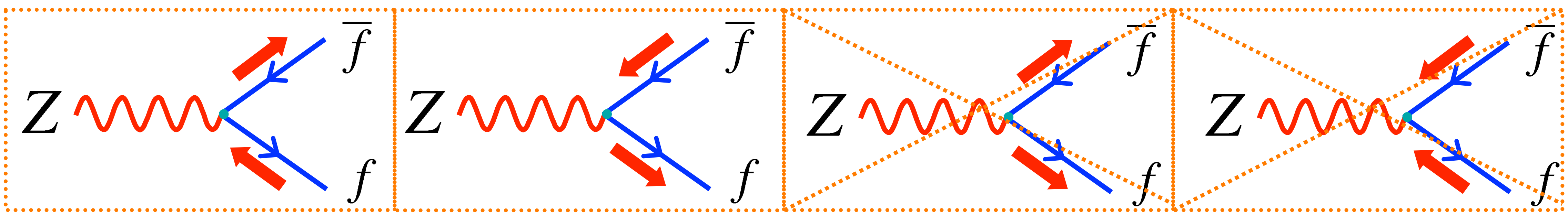
Z-boson decay

- ★ **In W-boson decay only had to consider one helicity combination of (assuming we can neglect final state masses: helicity states = chiral states)**



W-boson couples:
to LH particles
and RH anti-particles

- ★ **But Z-boson couples to LH and RH particles (with different strengths)**
- ★ **Need to consider only two helicity (or more correctly chiral) combinations:**



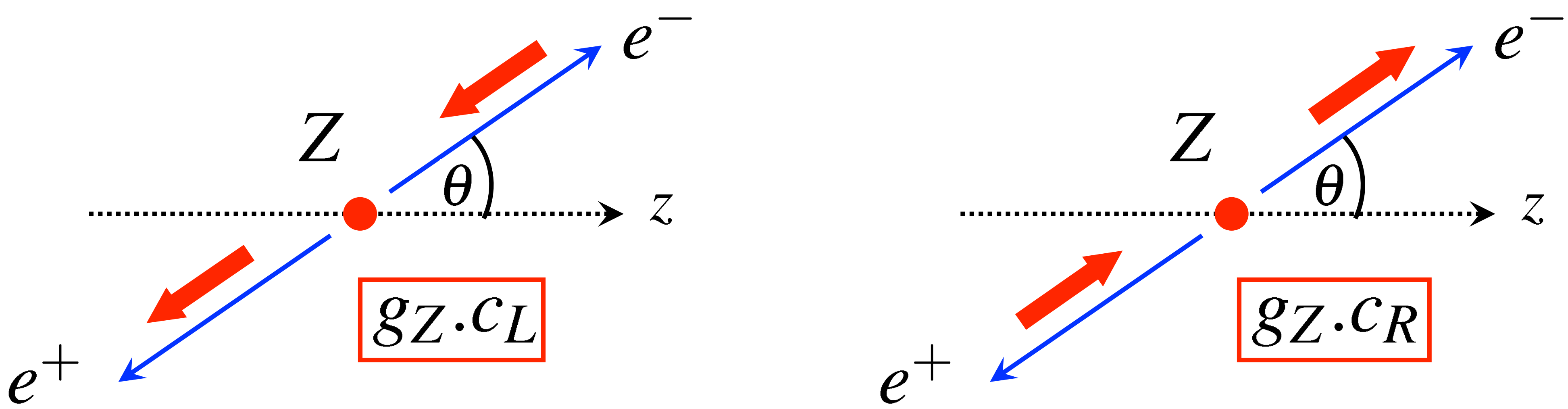
This can be seen by considering either of the combinations which give zero

e.g.

$$\overline{u}_R\gamma^\mu(c_V+c_A\gamma_5)v_R = u^\dagger\tfrac{1}{2}(1+\gamma^5)\gamma^0\gamma^\mu(c_V+c_A\gamma^5)\tfrac{1}{2}(1-\gamma^5)v$$
$$= \tfrac{1}{4}u^\dagger\gamma^0(1-\gamma^5)\gamma^\mu(1-\gamma^5)(c_V+c_A\gamma^5)v$$
$$= \tfrac{1}{4}\overline{u}\gamma^\mu(1+\gamma^5)(1-\gamma^5)(c_V+c_A\gamma_5)v = 0$$

Z-boson decay

★ **In terms of left and right-handed combinations need to calculate:**



★ **For unpolarized Z bosons:** **(Question 26)**

$$\langle |M_{fi}|^2 \rangle = \tfrac{1}{3}[2c_L^2 g_Z^2 m_Z^2 + 2c_R^2 g_Z^2 m_Z^2] = \tfrac{2}{3} g_Z^2 m_Z^2 (c_L^2 + c_R^2)$$

average over polarization

★ **Using** $c_V^2 + c_A^2 = 2(c_L^2 + c_R^2)$ **and** $\dfrac{\mathrm{d}\Gamma}{\mathrm{d}\Omega} = \dfrac{|p^*|}{32\pi^2 m_W^2}|M|^2$

$$\boxed{\Gamma(Z \to e^+e^-) = \frac{g_Z^2 m_Z}{48\pi}(c_V^2 + c_A^2)}$$

Z decay BRs

★ **(Neglecting fermion masses) obtain the same expression for the other decays**

$$\Gamma(Z \to f\bar{f}) = \frac{g_Z^2 m_Z}{48\pi}(c_V^2 + c_A^2)$$

• **Using values for c_V and c_A on page 471 obtain:**

$$Br(Z \to e^+e^-) = Br(Z \to \mu^+\mu^-) = Br(Z \to \tau^+\tau^-) \boxed{\approx 3.5\%}$$

$$Br(Z \to \nu_1\bar{\nu}_1) = Br(Z \to \nu_2\bar{\nu}_2) = Br(Z \to \nu_3\bar{\nu}_3) \boxed{\approx 6.9\%}$$

$$Br(Z \to d\bar{d}) = Br(Z \to s\bar{s}) = Br(Z \to b\bar{b}) \boxed{\approx 15\%}$$

$$Br(Z \to u\bar{u}) = Br(Z \to c\bar{c}) \boxed{\approx 12\%}$$

• **The Z Boson therefore predominantly decays to hadrons**

$$Br(Z \to \text{hadrons}) \approx 69\%$$

Mainly due to factor 3 from colour

• **Also predict total decay rate (total width)**

$$\Gamma_Z = \sum_i \Gamma_i = 2.5\,\text{GeV}$$

Experiment: $$\boxed{\Gamma_Z = 2.4952 \pm 0.0023\,\text{GeV}}$$

Summary

★ The Standard Model interactions are mediated by spin-1 gauge bosons

★ The form of the interactions are completely specified by the assuming an underlying local phase transformation ➡ GAUGE INVARIANCE

$\boxed{U(1)_{em}} \Rightarrow \boxed{\text{QED}}$

$\boxed{SU(2)_{L}} \Rightarrow \boxed{\text{Charged Current Weak Interaction} + W^3}$

$\boxed{SU(3)_{col}} \Rightarrow \boxed{\text{QCD}}$

★ In order to "unify" the electromagnetic and weak interactions, introduced a new symmetry gauge symmetry : U(1) hypercharge

$\boxed{U(1)_{Y}} \Rightarrow \boxed{B_{\mu}}$

★ The physical Z boson and the photon are mixtures of the neutral W boson and B determined by the Weak Mixing angle

$$\boxed{\sin\theta_W \approx 0.23}$$

★ Have we really unified the EM and Weak interactions ? Well not really…

- Started with two independent theories with coupling constants g_W, e
- Ended up with coupling constants which are related but at the cost of introducing a new parameter in the Standard Model θ_W
- Interactions not unified from any higher theoretical principle… but it works!