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General Relativity
Sheet 2

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Exercise 1 [Action principle for a charged relativistic particle]

Derive the equation of motion for a particle with mass m and charge e in an electromagnetic field given by the vector potential $A^\mu = (\phi, A^i)$ by applying the principle of least action on the action

$$S = -m \int ds \; - e \int A_\mu dx^\mu. \tag{1}$$

We use here units where $c = 1$. Write the result in terms of

a) the electromagnetic field tensor,

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}. \tag{2}$$

b) the electric and magnetic fields,

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla\phi, \qquad \mathbf{B} = \mathrm{rot}\,\mathbf{A}. \tag{3}$$

Exercise 2 [Relativistic motion in the Coulomb potential]

Now specialize to the case of a Coulomb potential using the gauge

$$A^\mu = (V(r), 0, 0, 0)\,,$$

where $V(r) = \frac{\alpha}{r}$, and set the charge of the particle to unity.

a) Before investigating the motion, it is best to make use of the symmetries of the problem. To this end, express the line element

$$ds = \sqrt{dt^2 - dx^2 - dy^2 - dz^2} \tag{4}$$

in spherical coordinates, i.e. $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$. Due to rotational symmetry, the motion is always restricted to a plane which we may choose to be the $x-y$-plane, so that in the following we set $\theta = \pi/2$. Write down the metric tensor in cartesian coordinates and in spherical coordinates.

Now show that the action dS accumulated during the passage through an infitesimal part of the particles' worldline is given by

$$dS = -\left(m\gamma + V(r)\right) dt + m\gamma\, \dot{r}\, dr + m\gamma r^2\, \dot{\phi}\, d\phi, \tag{5}$$

where $\dot{} = \frac{\mathrm{d}}{\mathrm{d}t}$ is the derivative by the time coordinate. Do this by carrying out the variation δS, applying partial integration and keeping boundary terms. The particle's trajectory can then be extended by an infinitesimal part $dt, dr, d\phi$ to obtain dS. Recall from classical mechanics that dS may be identified with physical quantities as

$$dS = -Edt + p_r dr + Ld\phi \tag{6}$$

and that symmetry under time-translations leads to conservation of the energy E, and that invariance under rotations around the z-axis leads to conservation of the z-component of the angular momentum L, and thus conclude that

$$\begin{aligned} L &= m\gamma r^2 \dot{\phi}, & (7) \\ (E - V(r))^2 &= m^2 + p_r^2 + L^2/r^2. & (8) \end{aligned}$$

Verify the conservation of L from the equation of motion.

b) The equation for the energy can be used to find an 'effective potential' for the radial coordinate for a given energy E and angular momentum L. Rewrite Eq. (8) in the form

$$p_r = \pm\sqrt{\tilde{E} - \tilde{U}(r)}, \tag{9}$$

where $\tilde{E} = E^2 - m^2$, and $\tilde{U}(r)$ contains the remaining terms. Because p_r is always real, the effective potential can be used to graphically discuss the behavior of the solution. Assuming it to be attractive ($\alpha < 0$), plot the potential for different parameters L and $\tilde{E}$ and discuss the qualitative features of the motion: Bound states vs. scattering, high vs. low angular momenta.

The function $S(t, r, \phi)$ is the basic object of the Hamilton-Jacobi approach to classical mechanics that is frequently used to investigate celestial dynamics.