Exercise 1. *Topological charge*

Consider a $SU(N)$ gauge theory with covariant derivative defined as

$$D_\mu = \partial_\mu + igA_\mu, \quad A_\mu = A_\mu^a T^a, \quad F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g f^{abc} A_\mu^b A_\nu^c \,, \tag{1}$$

where the T^a are the generators of $SU(N)$, which satisfy $[T^a, T^b] = i f^{abc} T^c$.

1. *Show that $F_{\mu\nu}^a \tilde{F}^{\mu\nu,a} = \frac{1}{2}\,\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^a F_{\rho\sigma}^a$ can be written as a total derivative, namely*

$$F_{\mu\nu}^a \tilde{F}^{\mu\nu,a} = \partial_\mu K^\mu \,, \qquad K^\mu = \epsilon^{\mu\nu\rho\sigma}\left(A_\nu^a F_{\rho\sigma}^a + \frac{g}{3} f_{abc} A_\nu^a A_\rho^b A_\sigma^c\right) . \tag{2}$$

2. *The topological charge is defined as*

$$n = \frac{g^2}{32\pi^2}\int \mathrm{d}^4x\, F_{\mu\nu}^a \tilde{F}^{\mu\nu,a} = \frac{g^2}{32\pi^2}\int \mathrm{d}^3x\, K_0\Big|_{t=-\infty}^{t=+\infty} . \tag{3}$$

Consider an adiabatic transformation of the type

$$\begin{aligned} A_\mu(t=-\infty) &= 0 \\ A_\mu(t=+\infty) &= \frac{i}{g}\,(\partial_\mu\Lambda)\,\Lambda^{-1}\,, \qquad \text{with} \qquad \Lambda = \frac{\mathbf{x}^2-d^2}{\mathbf{x}^2+d^2}\mathbf{1} + i\frac{2d}{\mathbf{x}^2+d^2}x_k\tau_k\,, \end{aligned} \tag{4}$$

where τ_i are the generators of a $SU(2)$ subgroup, $\tau_i\tau_j = \delta_{ij}\mathbf{1} + i\epsilon_{ijk}\tau_k$.
Check that $F_{\mu\nu} = 0$ and, hence, that the topological charge becomes

$$n = i\frac{g^3}{24\pi^2}\int \mathrm{d}^3x\,\epsilon^{ijk}\,\mathrm{Tr}\,\mathbf{A}_i\mathbf{A}_j\mathbf{A}_k\,. \tag{5}$$

3. *Show that $A_0(t=\infty) = 0$ and*

$$A_i(t=+\infty) = \frac{-2\,d}{g(\mathbf{x}^2+d^2)^2}\left[(\mathbf{x}^2-d^2)\tau_i - 2(x_j\tau_j)x_i + 2d\,\epsilon_{ijk}x_j\tau_k\right] . \tag{6}$$

4. *Compute the topological charge n of this adiabatic transformation.*
(Hint: recall the contraction identities $\epsilon^{ijk}\epsilon^{ijm} = 2\delta^{km}$ and $\epsilon^{ijk}\epsilon^{iml} = \delta^{jm}\delta^{kl} - \delta^{jl}\delta^{km}$).

Solution.

1. We start by writing the contraction of the field-strength tensor with its dual in terms of A_μ,

$$\begin{aligned} \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}^a F_{\rho\sigma}^a =& \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\epsilon^{\mu\nu\rho\sigma}\partial_\mu\left(A_\nu^a F_{\rho\sigma}^a + \frac{g}{3}f^{abc}A_\nu^a A_\rho^b A_\sigma^c\right)\left(\partial_\rho A_\sigma^a - \partial_\sigma A_\rho^a - g f^{ade}A_\rho^d A_\sigma^e\right) \\ =& \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\left(2\partial_\mu A_\nu^a - g f^{abc}A_\mu^b A_\nu^c\right)\left(2\partial_\rho A_\sigma^a - g f^{ade}A_\rho^d A_\sigma^e\right) \\ =& \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\Big[4(\partial_\mu A_\nu^a)(\partial_\rho A_\sigma^a) - 2g f^{ade}(\partial_\mu A_\nu^a)A_\rho^d A_\sigma^e - 2g f^{abc}(\partial_\rho A_\sigma^a)A_\mu^b A_\nu^c \end{aligned} \tag{S.1}$$

$$+g^2 f^{abc}f^{ade}A_\mu^b A_\nu^c A_\rho^d A_\sigma^e\Big] \,, \tag{S.2}$$

where, in the second equality, we have use the antisymmetry under the interchange of pair of Lorentz indices. The last term in brackets is symmetric under interchange of μ and ρ and, therefore, it vanishes in the contraction with the Levi-Civita tensor. By relabelling $d \to b$ and $e \to c$ and by interchanging $\mu \leftrightarrow \rho$ and $\nu \leftrightarrow \sigma$ in the third one, we see that the these two terms are identical. All in all, we obtain

$$\frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu} F^a_{\rho\sigma} = \epsilon^{\mu\nu\rho\sigma} \left(2(\partial_\mu A^a_\nu)(\partial_\rho A^a_\sigma) - 2g f^{abc} (\partial_\mu A^a_\nu) A^b_\rho A^c_\sigma\right) . \tag{S.3}$$

In a similar way, we can express the total derivative of K^μ in terms of the gauge field A_μ,

$$\begin{aligned}
\partial_\mu K^\mu =& \epsilon^{\mu\nu\rho\sigma} \partial_\mu \left(A^a_\nu F^a_{\rho\sigma} + \frac{g}{3} f^{abc} A^a_\nu A^b_\rho A^c_\sigma \right) \\
=& \epsilon^{\mu\nu\rho\sigma} \partial_\mu \left(A^a_\nu \left(2\partial_\rho A^a_\sigma - g f^{abc} A^b_\rho A^c_\nu \sigma \right) + \frac{g}{3} f^{abc} A^a_\nu A^b_\rho A^c_\sigma \right) \\
=& \epsilon^{\mu\nu\rho\sigma} \Big[2\partial_\mu A^a_\nu \partial_\rho A^a_\sigma + 2 A^a_\nu \partial_\mu \partial_\rho A^a_\sigma \\
& - \frac{2}{3} g f^{abc} \left((\partial_\mu A^a_\nu) A^b_\rho A^c_\sigma + A^a_\nu (\partial_\mu A^b_\rho) A^c_\sigma + A^a_\nu A^b_\rho (\partial_\mu A^c_\sigma) \right) \Big] .
\end{aligned} \tag{S.4}$$

In the contraction, the second derivative of eq. (S.4) vanishes and, by suitable relabelling of the Lorentz and colour indices, we can verify that the three terms $\sim g$ give identical contributions. Hence, we have

$$\partial_\mu K^\mu = \epsilon^{\mu\nu\rho\sigma} \left(2(\partial_\mu A^a_\nu)(\partial_\rho A^a_\sigma) - 2g f^{abc} (\partial_\mu A^a_\nu) A^b_\rho A^c_\sigma\right) . \tag{S.5}$$

By comparison of eq.s (S.3)-(S.5), we obtain

$$\frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu} F^a_{\rho\sigma} = \partial_\mu K^\mu . \tag{S.6}$$

2. For $t \to -\infty$, $F_{\mu\nu} = 0$ is obviously zero. For $t \to \infty$, we can use

$$\partial_\mu (\Lambda \Lambda^{-1}) \Lambda \partial_\mu \Lambda^{-1} + (\partial_\mu \Lambda) \Lambda^{-1} = 0$$

in order to show that

$$\begin{aligned}
-ig \left[A_\mu, A_\nu\right] &= \frac{i}{g} \left((\partial_\mu \Lambda)\Lambda^{-1} (\partial_\nu \Lambda)\Lambda^{-1} - (\partial_\nu \Lambda)\Lambda^{-1} (\partial_\mu \Lambda)\Lambda^{-1} \right) \\
&= \frac{i}{g} \left(-(\partial_\mu \Lambda)\Lambda^{-1}\Lambda(\partial_\nu \Lambda^{-1}) + (\partial_\nu \Lambda)\Lambda^{-1}\Lambda(\partial_\nu \Lambda^{-1}) \right) \\
&= \frac{i}{g} \left((\partial_\nu \Lambda)(\partial_\mu \Lambda^{-1}) - (\partial_\mu \Lambda)(\partial_\nu \Lambda^{-1}) \right) .
\end{aligned} \tag{S.7}$$

In addition, we have

$$\begin{aligned}
\partial_\mu A_\nu - \partial_\nu A_\mu &= \frac{i}{g} \left(\partial_\mu \left((\partial_\nu \Lambda)\Lambda^{-1} \right) - \partial_\nu \left((\partial_\mu \Lambda)\Lambda^{-1} \right) \right) \\
&= \frac{i}{g} \left((\partial_\mu \partial_\nu \Lambda)\Lambda^{-1} + (\partial_\nu \Lambda)(\partial_\mu \Lambda^{-1}) - (\partial_\nu \partial_\mu \Lambda)\Lambda^{-1} - (\partial_\mu \Lambda)(\partial_\nu \Lambda^{-1}) \right) \\
&= \frac{i}{g} \left((\partial_\nu \Lambda)(\partial_\mu \Lambda^{-1}) - (\partial_\mu \Lambda)(\partial_\nu \Lambda^{-1}) \right) .
\end{aligned} \tag{S.8}$$

Therefore, by combining eq.s (S.7)-(S.8), we obtain

$$F_{\mu\nu}|_{t\to\infty} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig \left[A_\mu, A_\nu\right] = 0. \tag{S.9}$$

The topological charge associate to this adiabatic transformation becomes

$$
\begin{aligned}
n =& \frac{g^2}{32\pi^2}\int \mathrm{d}^3x\, K_0\Big|_{t=-\infty}^{t=+\infty} = \frac{g^2}{32\pi^2}\int \mathrm{d}^3x\, K_0(t\to+\infty) = \\
=& \frac{g^2}{96\pi^2}\int \mathrm{d}^3x \epsilon^{0ijk} f^{abc} A_i^a A_j^b A_k^c = \frac{g^2}{96\pi^2}\int \mathrm{d}^3x \epsilon^{ijk} f^{abc} A_i^a A_j^b A_k^c \\
=& -\frac{ig^2}{48\pi^2}\int \mathrm{d}^3x \epsilon^{ijk}\,\mathrm{Tr}\left(T^aT^bT^c - T^bT^aT^c\right) A_i^a A_j^b A_k^c \\
=& -\frac{ig^2}{24\pi^2}\int \mathrm{d}^3x \epsilon^{ijk}\,\mathrm{Tr}\left(A_iA_jA_k\right) \\
=& \frac{ig^2}{24\pi^2}\int \mathrm{d}^3x \epsilon^{ijk}\,\mathrm{Tr}\left(\mathbf{A}_i\mathbf{A}_j\mathbf{A}_k\right) . \qquad \text{(S.10)}
\end{aligned}
$$

In the fifth equality, we have used the definition $f^{abc} = -2i\,\mathrm{Tr}\left[T^a, T^b\right] T^c$. In the last two equalities, we have used the definition $A_i = A_i^a T^a$ and we recalled that, for space-like components, $A_i = -\mathbf{A}_i$

3. The gauge transformation Λ does not depend on time and, therefore, $A_0(t=+\infty) = 0$. From the explicit expression of Λ, we can verify that

$$
\Lambda^{-1} = \frac{\mathbf{x}^2 - d^2}{\mathbf{x}^2 + d^2}\mathbf{1} - i\frac{2d}{\mathbf{x}^2 + d^2} x_k\tau_k \qquad \text{(S.11)}
$$

and that

$$
\partial_i\Lambda = \frac{2d}{(\mathbf{x}^2 + d^2)^2}\left(4dx_i\mathbf{1} + i(\mathbf{x}^2 + d^2)\tau_i - 2x_ix_k\tau_k\right) . \qquad \text{(S.12)}
$$

Starting from eq.s (S.11)-(S.12), we obtain

$$
\begin{aligned}
A_i(t=+\infty) =& \frac{i}{g}\left(\partial_i\Lambda\right)\Lambda^{-1} \\
=& \frac{i}{g}\frac{1}{(\mathbf{x}^2+d^2)^3}\left[x_i\left(-4id(\mathbf{x}^2+d^2)x_k\tau_k\right) + \tau_i\left(2id(\mathbf{x}^4 - d^4)\right) + \right. \\
& \left. i\epsilon_{ikr}x_k\tau_r 4\left(d^2(\mathbf{x}^2+d^2)\right)\right] \\
=& \frac{-2\,d}{g(\mathbf{x}^2+d^2)^2}\left[(\mathbf{x}^2 - d^2)\tau_i - 2(x_j\tau_j)x_i + 2d\,\epsilon_{ijk}x_j\tau_k\right] .
\end{aligned}
$$

4. The calculation of the trace in eq. (S.10) is rather tedious and it will not be reported here. Nevertheless it is sufficient to use the expression of $A_i(t=+\infty)$ previously derived, as well as the trace identities

$$
\mathrm{Tr}(\tau^i, \tau^j) = 2\delta^{ij} , \qquad \mathrm{Tr}(\tau^i\tau^j\tau^k) = 2i\epsilon^{ijk} \qquad \text{(S.13)}
$$

and the contraction identities

$$
\epsilon^{ijk}\epsilon^{ijm} = 2\delta^{km} , \qquad \epsilon^{ijk}\epsilon^{iml} = \delta^{jm}\delta^{kl} - \delta^{jl}\delta^{km}. \qquad \text{(S.14)}
$$

in order to arrive at

$$
n = \frac{4d^3}{\pi^2}\int \mathrm{d}^3x\,\frac{1}{(\mathbf{x}^2+d^2)^3} . \qquad \text{(S.15)}
$$

the space integral can be evaluated by moving to spherical coordinates,

$$
n = \frac{16d^3}{\pi}\int_0^\infty \mathrm{d}r\,\frac{r^2}{(r^2+d^2)^3} = \frac{8d^3}{\pi}\int_{-\infty}^\infty \mathrm{d}r\,\frac{r^2}{(r^2+d^2)^3} . \qquad \text{(S.16)}
$$

Since the integrand behaves like $\sim 1/r^4$ for $r \to \infty$, the radial integral can be computed with Cauchy theorem,

$$\begin{aligned} n =& \frac{8d^3}{\pi} \lim_{R\to\infty} \int_{-R}^{R} \mathrm{d}r \, \frac{r^2}{(r^2+d^2)^3} \\ =& \frac{8d^3}{\pi} \lim_{R\to\infty} \left(\int_{-R}^{R} \mathrm{d}r \, \frac{r^2}{(r^2+d^2)^3} + \int_{\gamma_R} \mathrm{d}r \, \frac{r^2}{(r^2+d^2)^3} \right) \, , \end{aligned} \tag{S.17}$$

with γ_R being a semi-circle in the upper-plane centred in the origin. In this way, we obtain

$$n = \frac{8d^3}{\pi} \mathrm{Res} \left[\frac{r^2}{(r-id)^3(r+id)^3} \right] \Bigg|_{r=id} = 8id^3 \lim_{r\to id} \frac{d^2}{dr^2} \frac{r^2}{(r+id)^3} = 1 \, . \tag{S.18}$$