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General Relativity
Sheet 8

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Exercise 1 [Divergence and Current Conservation in Curved Spacetime]

In this exercise, you will show that the current conservation law known from electrodynamics in flat spacetime generalizes to curved spacetimes. The main difference is that partial derivatives ∂_μ have to be replaced by covariant derivatives ∇_μ that are determined by the Christoffel symbols

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left(g_{\mu\sigma,\nu} + g_{\nu\sigma,\mu} - g_{\mu\nu,\sigma} \right).$$

a) Show that the covariant divergence of a vector field $\mathbf{J}$, defined by $\nabla \cdot \mathbf{J} \equiv \nabla_\mu J^\mu = J^\mu{}_{;\mu}$ can be expressed in terms of the flat spacetime divergence $V^\mu{}_{,\mu}$ of another vector field $\mathbf{V} = \sqrt{|\det g|}\mathbf{J}$.
Hint: Use the identity $e^{\mathrm{tr}A} = \det e^A$*, where* A *is a matrix, to compute the derivative of* $\sqrt{|\det g|}$.

b) Find the conditions that the tensor field $\mathbf{F} = F^{\mu\nu}\, \partial_\mu \otimes \partial_\nu$ needs to satisfy to fulfil the expression

$$F^{\mu\nu}{}_{;\nu} = \frac{1}{\sqrt{|\det g|}} \left(\sqrt{|\det g|} F^{\mu\nu} \right)_{,\nu}.$$

c) Conclude from the above that Maxwell's equations generalized to a curved spacetime, i.e. $F^{\mu\nu}{}_{;\nu} = -4\pi J^\mu$, imply the current conservation law $\nabla \cdot \mathbf{J} = 0$.

Exercise 2 [Geometry and the equivalence principle]

Prove that we can define a locally inertial coordinate system at a point P and that the deviations from flatness come at second order in Taylor expansion. Start with an arbitrary (unprimed) coordinate system with arbitrary metric and look for a coordinate transformation that will transform it into the desired (primed) coordinate system.

Expand the coordinate transformation and the metric in Taylor series around the point P to second order in primed coordinates and insert this into the relation between the metric in primed and unprimed coordinates, grouping the terms at the same order of Taylor expansion.

a) At zeroth order, count the number of equations and number of free variables in the transformation. Can you make the primed metric at P equal to the Minkowski metric? If yes, how many extra variables do you have and what degrees of freedom do they correspond to?

b) Next, we want the metric elements in the primed system to vanish at first order. How many free variables of the tranformation and how many metric elements are there at this order, keeping in mind the symmetries? Can we satisfy the equations?

c) Same as above at second order. Can we satisfy the equations this time? If not, how many more equations than free variables do we have?

Exercise 3 [Gravitational Stress and the Lie Derivative]

In this exercise you use the Lie derivative to analyze the deformation of a dust cloud in a gravitational field. The motion of the cloud is described by the four-velocity vector *field* $\mathbf{u}$ that describes the motion of each particle in the cloud. Clearly, $\mathbf{u}$ contains a lot of information which does not give us explicit information about what happens in the cloud. To make sense of the physical content of $\mathbf{u}$, it is useful to form tensorial quantities that directly carry physical information. More concretely, consider $\nabla \cdot \mathbf{u}$. You will establish that this quantity is connected to changes in volume of the cloud.

a) First consider a local Lorentz frame that is comoving with the particles at some point P. Thus, in this frame and for a small volume around P, you can set $g_{\mu\nu} = \eta_{\mu\nu}$. Moreover, the four-velocity field in the vicinity of P can be approximated by $\mathbf{u}(x) = \partial_t + v^i(x)\partial_i$, as the neighboring particles are expected to move with a similar (at least non-relativistic) speed relative to the particle in P, so the factor $\gamma \approx 1$.

   Let ϕ_τ be the flow associated to $\mathbf{u}$, i.e. $\mathbf{u} = \frac{d}{d\tau}\phi_\tau$. Let $\mathrm{vol}_3 = dx \wedge dy \wedge dz$ be the spatial volume form in the Lorentz frame and denote by $R(0)$ a (small) cubic spacetime region with volume $V(0) = \Delta x \Delta y \Delta z = L^3$ initially occupied by the particles. Use the pullback of vol_3 w.r.t. ϕ_τ to show that the rate of change of the volume is given by

$$\frac{d}{d\tau}V(\tau)\Big|_{\tau=0} = \int_{R(0)} \mathcal{L}_{\mathbf{u}}\,\mathrm{vol}_3,$$

   where $\mathcal{L}$ is the Lie derivative. Explicitly calculate the Lie derivative and express the result in terms of $\nabla \cdot \mathbf{u} = \nabla_\mu u^\mu$, and thus conclude that it measures the logarithmic change in spatial volume with proper time. *Hint: Remember that the integral is done over a region at constant time.*

b) The calculation in part a) is valid locally for infinitesimally small volumes. Now we extend this result to finite spatial regions in the weak-field, small-velocity limit.

   We use the weak-field metric for a static source,

$$ds^2 = -(1+2\psi)\,dt^2 + (1-2\psi)\left(dx^2+dy^2+dz^2\right),$$

   where ψ depends on position, but not on time. In this case, because there are no terms that couple space- and time-directions, it makes sense to decompose spacetime into spatial slices, one for each time t, and each endowed with spatial metric

$$\gamma_{ij} = (1-2\psi)\,\delta_{ij}dx^i \otimes dx^j,$$

   and associated volume three-form $\mathrm{vol}_3 = \sqrt{\det\gamma}\,dx \wedge dy \wedge dz$ (at this point we have chosen one particular family of observers). Moreover, we restrict ourselves to the non-relativistic limit, i.e. $v^2 \ll 1$.

   Now we denote by $R(t)$ some finite subset of $\mathbb{R}^3$ at coordinate time t. The region evolves under the flow ϕ_t that corresponds to the four-velocity $\mathbf{u}$. Let

$$V(t) = \int_{R(t)} \mathrm{vol}_3$$

   be the time-dependent volume of the region. Evaluate the rate of change $\frac{d}{dt}V(t)$ of the volume of $R(t)$ due to the flow caused by $\mathbf{u}(t,\mathbf{x})$ and show that it is given by

$$\frac{d}{dt}V(t) = \int_{R(t)} \nabla\cdot\mathbf{v}\,\mathrm{vol}_3,$$

   where now ∇ is the three-dimensional covariant derivative (i.e. $\nabla\cdot\mathbf{v} = \nabla_i v^i$) and $\mathbf{v}$ is the space part of $\mathbf{u}$ (i.e. $\mathbf{v} = (u^1, u^2, u^3)$).