Universität Zürich[UZH]

Physik-Institut

Kern- und Teilchenphysik II
Lecture7: ep Scattering

(adapted from the Handout of Prof. Mark Thomson)

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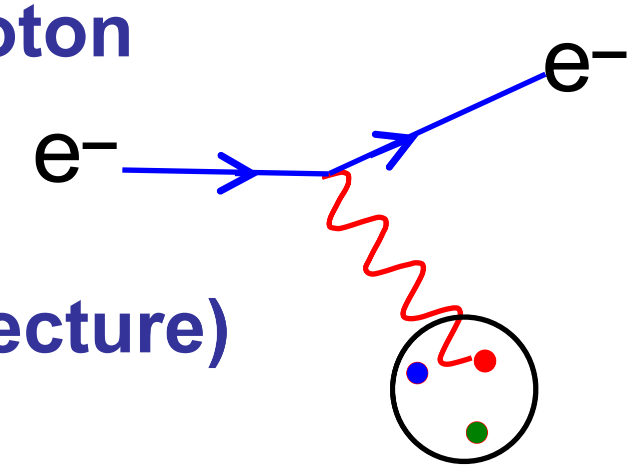
http://www.physik.uzh.ch/de/lehre/PHY213/FS2018.html

Electron-Proton Scattering

- In this handout aiming towards a study of electron-proton scattering as a probe of the structure of the proton
- Two main topics:
  - $e^- p \rightarrow e^- p$ elastic scattering
  - $e^- p \rightarrow e^- X$ deep inelastic scattering (next lecture)

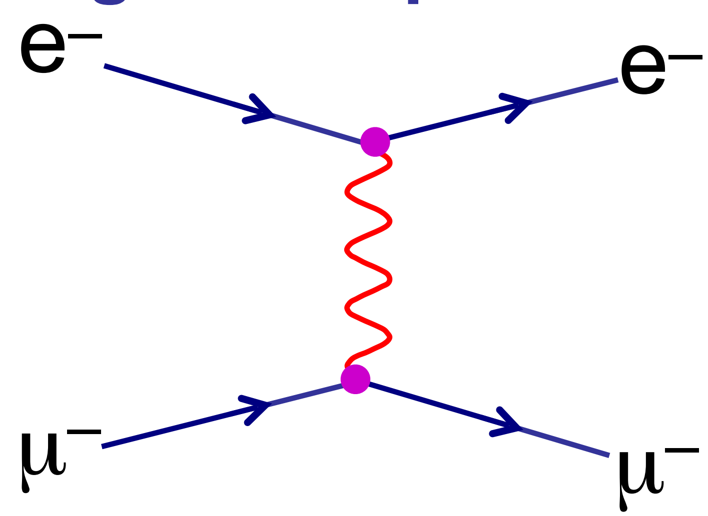


- But first consider scattering from a point-like particle e.g.

  $e^- \mu^- \rightarrow e^- \mu^-$

  i.e. the QED part of

  $(e^- q \rightarrow e^- q)$



- take results from $e^+e^- \rightarrow \mu^+\mu^-$ and use **"Crossing Symmetry"** to obtain the matrix element for $e^-\mu^- \rightarrow e^-\mu^-$ **(1)**

$$\langle |M_{fi}|^2 \rangle = 2e^4 \frac{(p_1.p_3)^2 + (p_1.p_4)^2}{(p_1.p_2)^2}$$

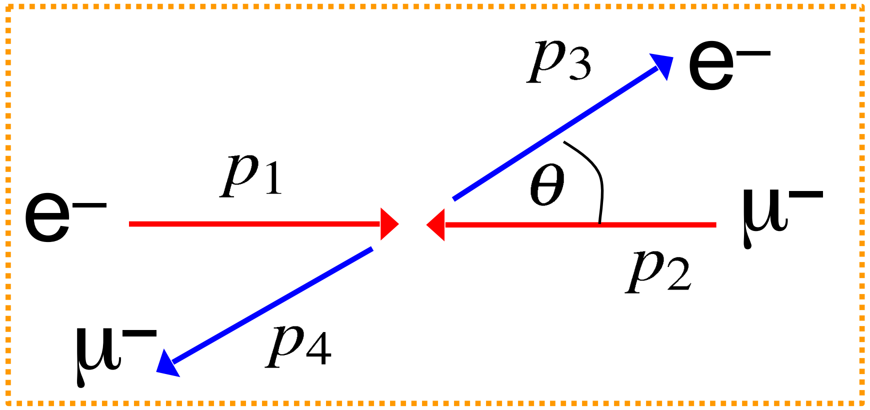
$$p_1 \rightarrow p_1,\; p_2 \rightarrow -p_3,\; p_3 \rightarrow p_4,\; p_4 \rightarrow -p_2$$

$$\langle|M_{fi}|^2\rangle = 2e^4 \frac{(p_1.p_4)^2+(p_1.p_2)^2}{(p_1.p_3)^2} \quad \textbf{(2)}$$

$$\equiv 2e^4\left(\frac{s^2+u^2}{t^2}\right)$$

Work in the C.o.M:

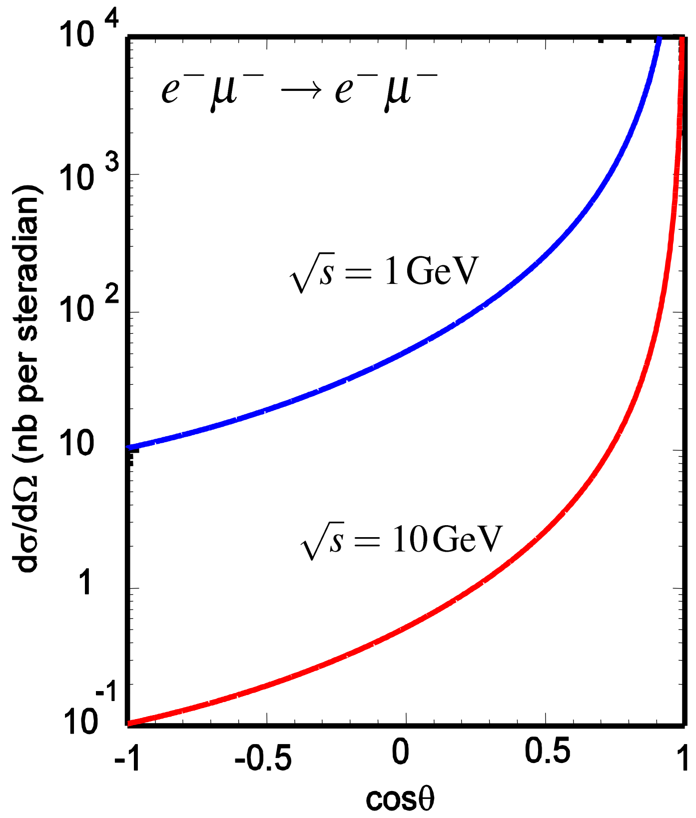
$$p_1 = (E,0,0,E) \qquad p_2 = (E,0,0,-E)$$
$$p_3 = (E, E\sin\theta, 0, E\cos\theta)$$
$$p_4 = (E, -E\sin\theta, 0, -E\cos\theta)$$



giving $p_1.p_2 = 2E^2; \quad p_1.p_3 = E^2(1-\cos\theta); \quad p_1.p_4 = E^2(1+\cos\theta)$

$$\rightarrow \quad \langle|M_{fi}|^2\rangle = 2e^4 \frac{E^4(1+\cos\theta)^2+4E^4}{E^4(1-\cos\theta)^2}$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\langle|M_{fi}|^2\rangle = \frac{e^4}{8\pi^2 s}\frac{\left[1+\frac{1}{4}(1+\cos\theta)^2\right]}{(1-\cos\theta)^2}$$



The <u>denominator</u> arises from the propagator $-ig_{\mu\nu}/q^2$

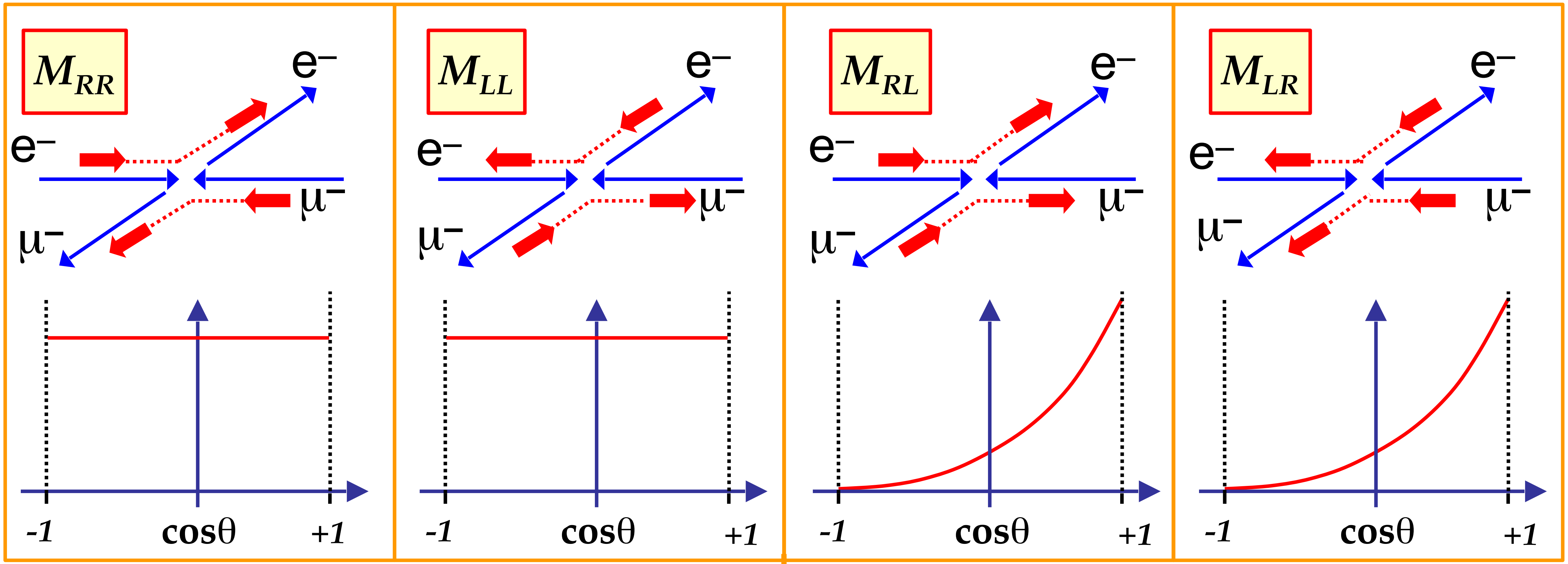
here $q^2 = (p_1-p_3)^2 = E^2(1-\cos\theta)$

as $q^2 \rightarrow 0$ **the cross section tends to infinity.**

- What about the angular dependence of the numerator ?

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{8\pi^2 s}\frac{\left[1+\frac{1}{4}(1+\cos\theta)^2\right]}{(1-\cos\theta)^2}$$

- The factor $1+\frac{1}{4}(1+\cos\theta)^2$ reflects helicity (really chiral) structure of QED
- Of the 16 possible helicity combinations only 4 are non-zero:



$S_z = 0$

$\rightarrow \frac{d\sigma}{d\Omega} \propto 1$

i.e. no preferred polar angle

$S_z = +1$ $\quad S_z = -1$

$\rightarrow \frac{d\sigma}{d\Omega} \propto \frac{1}{4}(1+\cos\theta)^2$

spin 1 rotation again

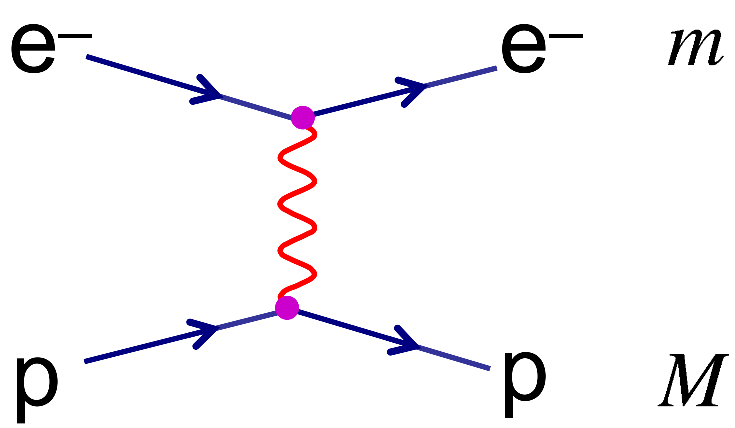
- The cross section calculated above is appropriate for the scattering of two spin half Dirac (i.e. point-like) particles in the ultra-relativistic limit (where both electron and muon masses can be neglected). In this case

$$\langle |M_{fi}|^2 \rangle = 2e^4 \frac{(p_1.p_4)^2 + (p_1.p_2)^2}{(p_1.p_3)^2}$$

- We will use this again in the discussion of "Deep Inelastic Scattering" of electrons from the quarks within a proton

- Before doing so we will consider the scattering of electrons from the composite proton - **i.e. how do we know the proton isn't fundamental "point-like" particle ?**

- In this discussion we will not be able to use the relativistic limit and require the general expression for the matrix element:

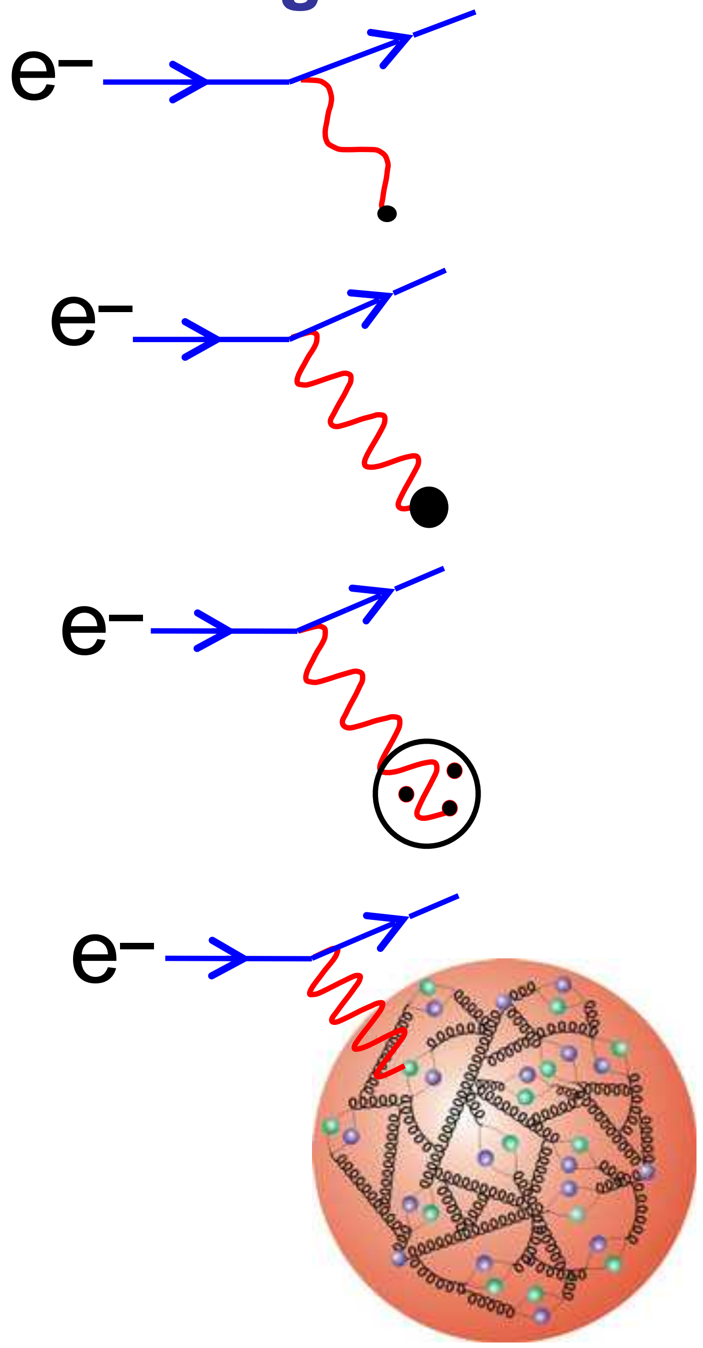


$$\langle |M_{fi}|^2 \rangle = \frac{8e^4}{(p_1 - p_3)^4} \left[(p_1.p_2)(p_3.p_4) + (p_1.p_4)(p_2.p_3) - (p_1.p_3)M^2 - (p_1.p_4)m^2 + 2m^2M^2 \right] \quad (3)$$

Probing the Structure of the Proton

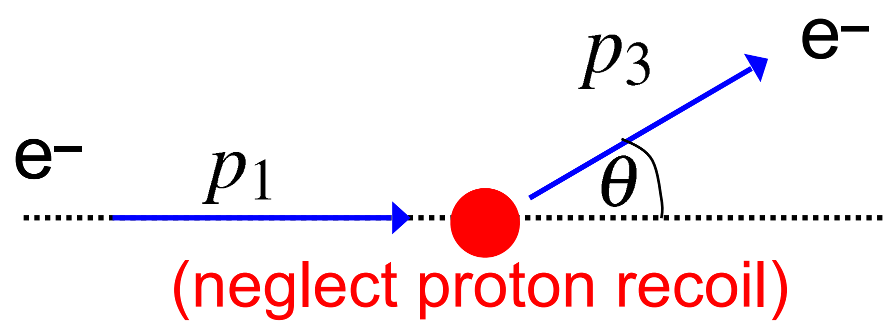
★In e⁻p → e⁻p scattering the nature of the interaction of the virtual photon with the proton depends strongly on wavelength

- ♦ At **very low** electron energies $\lambda \gg r_p$:
  the scattering is equivalent to that from a "point-like" **spin-less** object
- ♦ At **low** electron energies $\lambda \sim r_p$:
  the scattering is equivalent to that from a extended charged object
- ♦ At **high** electron energies $\lambda < r_p$:
  the wavelength is sufficiently short to resolve sub-structure. Scattering from constituent quarks
- ♦ At **very high** electron energies $\lambda \ll r_p$:
  the proton appears to be a sea of quarks and gluons.



Rutherford Scattering Revisited

★ Rutherford scattering is the **low energy limit** where the recoil of the proton can be neglected and the **electron is non-relativistic**



• Start from RH and LH Helicity particle spinors

$$u_\uparrow = N\begin{pmatrix} c \\ e^{i\phi}s \\ \frac{|\vec{p}|}{E+m}c \\ \frac{|\vec{p}|}{E+m}e^{i\phi}s \end{pmatrix} \quad u_\downarrow = N\begin{pmatrix} -s \\ e^{i\phi}c \\ \frac{|\vec{p}|}{E+m}s \\ -\frac{|\vec{p}|}{E+m}e^{i\phi}c \end{pmatrix}$$

$$N = \sqrt{E+m}; \qquad s = \sin(\theta/2); \quad c = \cos(\theta/2)$$

• Now write in terms of:

$$\alpha = \frac{|\vec{p}|}{E+m_e}$$

Non-relativistic limit: $\alpha \to 0$

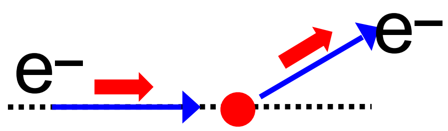
Ultra-relativistic limit: $\alpha \to 1$

$$\Rightarrow \quad u_\uparrow = N\begin{pmatrix} c \\ e^{i\phi}s \\ \alpha c \\ \alpha e^{i\phi}s \end{pmatrix} \quad u_\downarrow = N\begin{pmatrix} -s \\ e^{i\phi}c \\ \alpha s \\ -\alpha e^{i\phi}c \end{pmatrix}$$

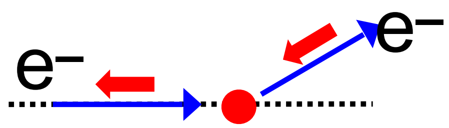
and the possible initial and final state electron spinors are:

$$u_\uparrow(p_1) = N_e\begin{pmatrix} 1 \\ 0 \\ \alpha \\ 0 \end{pmatrix} \quad u_\downarrow(p_1) = N_e\begin{pmatrix} 0 \\ 1 \\ 0 \\ -\alpha \end{pmatrix} \quad u_\uparrow(p_3) = N_e\begin{pmatrix} c \\ s \\ \alpha c \\ \alpha s \end{pmatrix} \quad u_\downarrow(p_3) = N_e\begin{pmatrix} -s \\ c \\ \alpha s \\ -\alpha c \end{pmatrix}$$

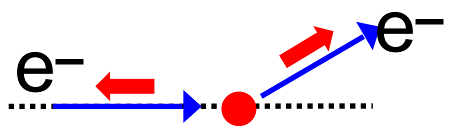
- Consider all four possible electron currents, i.e. Helicities **R→R, L→L, L→R, R→L**



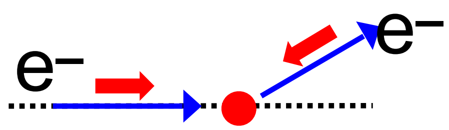
$$\bar{u}_\uparrow(p_3)\gamma^\mu u_\uparrow(p_1) = (E+m_e)\left[(\alpha^2+1)c, 2\alpha s, -2i\alpha s, 2\alpha c\right] \quad (4)$$



$$\bar{u}_\downarrow(p_3)\gamma^\mu u_\downarrow(p_1) = (E+m_e)\left[(\alpha^2+1)c, 2\alpha s, -2i\alpha s, 2\alpha c\right] \quad (5)$$



$$\bar{u}_\uparrow(p_3)\gamma^\mu u_\downarrow(p_1) = (E+m_e)\left[(1-\alpha^2)s, 0, 0, 0\right] \quad (6)$$



$$\bar{u}_\downarrow(p_3)\gamma^\mu u_\uparrow(p_1) = (E+m_e)\left[(\alpha^2-1)s, 0, 0, 0\right] \quad (7)$$

- In the relativistic limit ($\alpha = 1$), i.e. $E \gg m$

(6) and (7) are identically zero; only R→R and L→L combinations non-zero

- In the non-relativistic limit, $|\vec{p}| \ll E$ we have $\alpha = 0$

$$\bar{u}_\uparrow(p_3)\gamma^\mu u_\uparrow(p_1) = \bar{u}_\downarrow(p_3)\gamma^\mu u_\downarrow(p_1) = (2m_e)\left[c, 0, 0, 0\right]$$

$$\bar{u}_\uparrow(p_3)\gamma^\mu u_\downarrow(p_1) = -\bar{u}_\downarrow(p_3)\gamma^\mu u_\uparrow(p_1) = (2m_e)\left[s, 0, 0, 0\right]$$

All four electron helicity combinations have non-zero Matrix Element

i.e. Helicity eigenstates ≠ Chirality eigenstates

- The initial and final state **proton** spinors (assuming no recoil) are:

$$u_{\uparrow}(0) = \sqrt{2M_p}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \qquad u_{\downarrow}(0) = \sqrt{2M_p}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

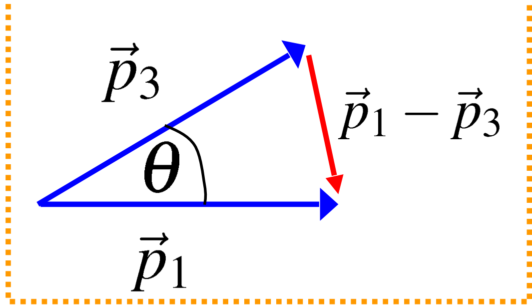
Solutions of Dirac equation for a particle at rest

giving the proton currents:

$$j_{p\uparrow\uparrow} = j_{p\downarrow\downarrow} = 2M_p\,(1,0,0,0)$$
$$j_{p\uparrow\downarrow} = j_{p\downarrow\uparrow} = 0$$

- The spin-averaged ME summing over the 8 allowed helicity states

$$\langle |M_{fi}^2| \rangle = \frac{1}{4}\frac{e^4}{q^4}(16M_p^2 m_e^2)(4c^2+4s^2) = \frac{16M_p^2 m_e^2 e^4}{q^4}$$



where $q^2 = (p_1 - p_3)^2 = (0, \vec{p}_1 - \vec{p}_3)^2 = -4|\vec{p}|^2 \sin^2(\theta/2)$

$$\langle |M_{fi}^2| \rangle = \frac{M_p^2 m_e^2 e^4}{|\vec{p}|^4 \sin^4(\theta/2)}$$

Note: in this limit all angular dependence is in the propagator

- The formula for the differential cross-section in the lab. frame was derived in KTI:

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{1}{M + E_1 - E_1\cos\theta}\right)^2 |M_{fi}|^2 \qquad (8)$$

- Here the electron is non-relativistic so $E \sim m_e \ll M_p$ and we can neglect E_1 in the denominator of equation (8)

$$\rightarrow \quad \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2 M_p^2}|M_{fi}|^2 = \frac{m_e^2 e^4}{64\pi^2 |\vec{p}|^4 \sin^4(\theta/2)}$$

- Writing $e^2 = 4\pi\alpha$ and the kinetic energy of the electron as $E_K = p^2/2m_e$

$$\rightarrow \quad \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{Rutherford}} = \frac{\alpha^2}{16 E_K^2 \sin^4\theta/2} \qquad (9)$$

★ This is the normal expression for the Rutherford cross section. It could have been derived by considering the scattering of a non-relativistic particle in the static Coulomb potential of the proton $V(\vec{r})$, without any consideration of the interaction due to the intrinsic magnetic moments of the electron or proton. From this we can conclude, that in this non-relativistic limit only the interaction between the electric charges of the particles matters.

The Mott Scattering Cross Section

- For Rutherford scattering we are in the limit where the target recoil is neglected and the scattered particle is non-relativistic $E_K \ll m_e$
- The limit where the target recoil is neglected and the scattered particle is **relativistic** (i.e. just neglect the electron mass) is called Mott Scattering
- In this limit the electron currents, equations (4) and (6), become:

$$\overline{u}_{\uparrow}(p_3)\gamma^{\mu}u_{\uparrow}(p_1) = 2E\,[c,s,-is,c] \qquad \overline{u}_{\uparrow}(p_3)\gamma^{\mu}u_{\downarrow}(p_1) = E\,[0,0,0,0]$$

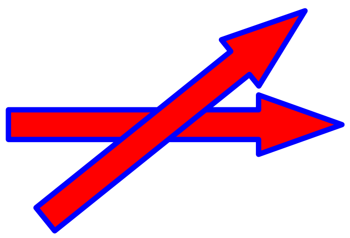
Relativistic ➡ Electron "helicity conserved"

- It is then straightforward to obtain the result:

$$\rightarrow \left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} = \frac{\alpha^2}{4E^2\sin^4\theta/2}\cos^2\frac{\theta}{2} \qquad (10)$$

Rutherford formula with $E_K = E \quad (E \gg m_e)$

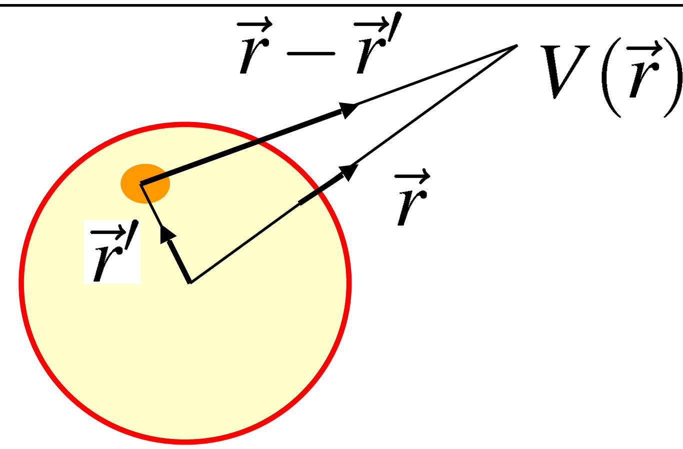
Overlap between initial/final state electron wave-functions. Just QM of spin ½



★ NOTE: we could have derived this expression from scattering of electrons in a static potential from a fixed point in space $V(\vec{r})$.
The interaction is **ELECTRIC** rather than magnetic (spin-spin) in nature.

★ Still haven't taken into account the charge distribution of the proton…..

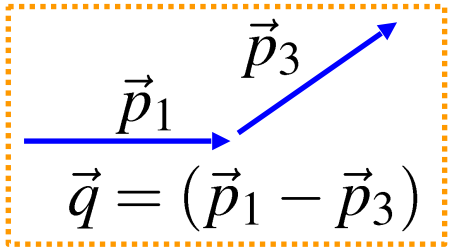
Form Factors



- Consider the scattering of an electron in the static potential due to an extended charge distribution.
- The potential at $\vec{r}$ from the centre is given by:

$$V(\vec{r}) = \int \frac{Q\rho(\vec{r}')}{4\pi|\vec{r}-\vec{r}'|} d^3\vec{r}' \quad \text{with} \quad \int \rho(\vec{r})\mathrm{d}^3\vec{r} = 1$$

- In first order perturbation theory the matrix element is given by:



$$M_{fi} = \langle \psi_f | V(\vec{r}) | \psi_i \rangle = \int e^{-i\vec{p}_3.\vec{r}} V(\vec{r}) e^{i\vec{p}_1.\vec{r}} \mathrm{d}^3\vec{r}$$

$$= \int\int e^{i\vec{q}.\vec{r}} \frac{Q\rho(\vec{r}')}{4\pi|\vec{r}-\vec{r}'|} \mathrm{d}^3\vec{r}'\mathrm{d}^3\vec{r} = \int\int e^{i\vec{q}.(\vec{r}-\vec{r}')} e^{i\vec{q}.\vec{r}'} \frac{Q\rho(\vec{r}')}{4\pi|\vec{r}-\vec{r}'|} \mathrm{d}^3\vec{r}'\mathrm{d}^3\vec{r}$$

- Fix $\vec{r}'$ and integrate over $\mathrm{d}^3\vec{r}$ with substitution $\vec{R} = \vec{r} - \vec{r}'$

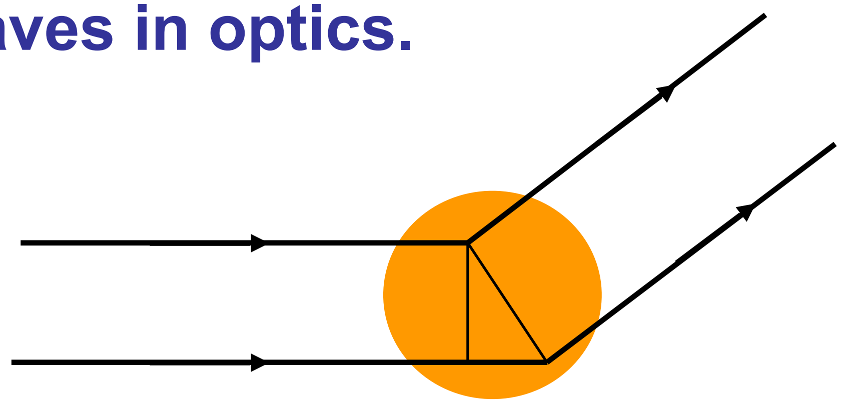
$$M_{fi} = \int e^{i\vec{q}.\vec{R}} \frac{Q}{4\pi|\vec{R}|} \mathrm{d}^3\vec{R} \int \rho(\vec{r}') e^{i\vec{q}.\vec{r}'} \mathrm{d}^3\vec{r}' = (M_{fi})_{point} F(\vec{q}^2)$$

★ The resulting matrix element is equivalent to the matrix element for scattering from a **point source** multiplied by the **form factor**

$$F(\vec{q}^2) = \int \rho(\vec{r}) e^{i\vec{q}.\vec{r}} \mathrm{d}^3\vec{r}$$

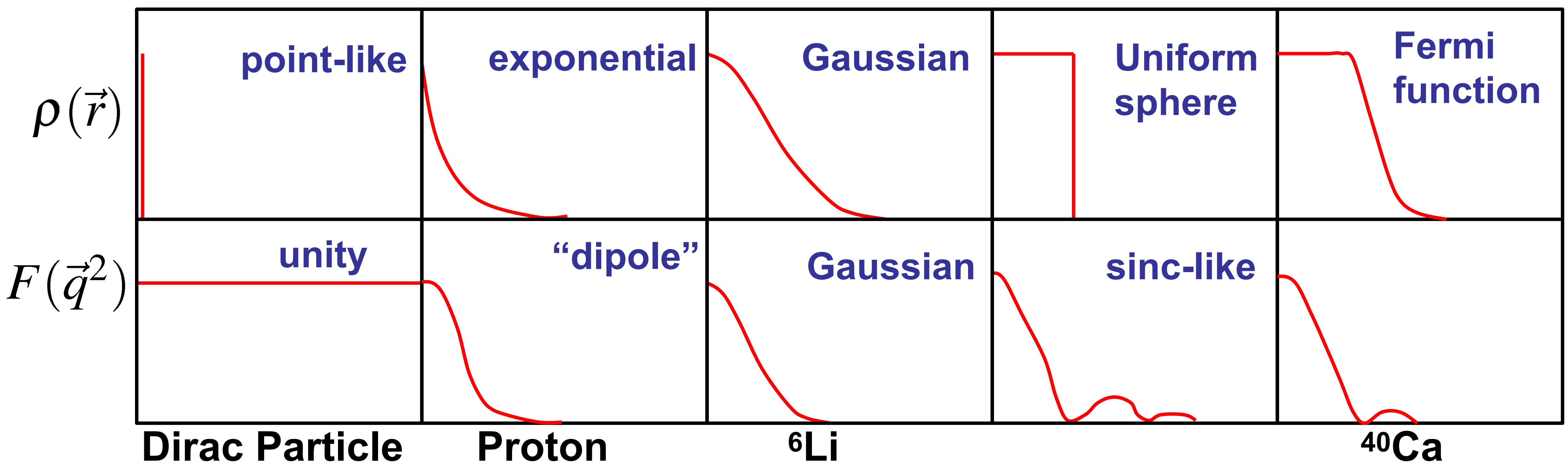
$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} \rightarrow \frac{\alpha^2}{4E^2\sin^4\theta/2}\cos^2\frac{\theta}{2}\left|F(\vec{q}^2)\right|^2$$

- There is nothing mysterious about form factors – similar to diffraction of plane waves in optics.



- The finite size of the scattering centre introduces a phase difference between plane waves "scattered from different points in space". If wavelength is long compared to size all waves in phase and $F(\vec{q}^2) = 1$

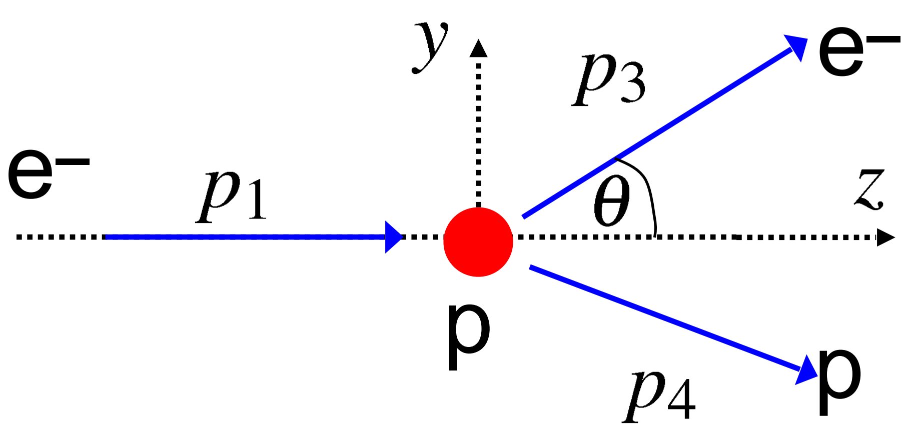
For example:



- **NOTE that for a point charge the form factor is unity.**

Point-like Electron-Proton Elastic Scattering

- So far have only considered the case we the proton does not recoil... For $E_1 \gg m_e$ the general case is



$$p_1 = (E_1, 0, 0, E_1)$$
$$p_2 = (M, 0, 0, 0)$$
$$p_3 = (E_3, 0, E_3 \sin\theta, E_3 \cos\theta)$$
$$p_4 = (E_4, \vec{p}_4)$$

- From Eqn. (2) with $m = m_e = 0$ the matrix element for this process is:

$$\langle |M_{fi}|^2 \rangle = \frac{8e^4}{(p_1 - p_3)^4} \left[(p_1.p_2)(p_3.p_4) + (p_1.p_4)(p_2.p_3) - (p_1.p_3)M^2 \right] \quad \textbf{(11)}$$

- Experimentally observe scattered electron so eliminate p_4
- The scalar products not involving p_4 are:

$$p_1.p_2 = E_1 M \qquad p_1.p_3 = E_1 E_3 (1 - \cos\theta) \qquad p_2.p_3 = E_3 M$$

- From momentum conservation can eliminate p_4 : $\quad p_4 = p_1 + p_2 - p_3$

$$p_3.p_4 = p_3.p_1 + p_3.p_2 - p_3.p_3 = E_1 E_3 (1 - \cos\theta) + E_3 M$$
$$p_1.p_4 = p_1.p_1 + p_1.p_2 - p_1.p_3 = E_1 M - E_1 E_3 (1 - \cos\theta)$$

$$p_1.p_1 = E_1^2 - |\vec{p}_1|^2 = m_e^2 \approx 0$$ i.e. neglect m_e

• Substituting these scalar products in Eqn. (11) gives

$$\begin{aligned}\langle |M_{fi}|^2\rangle &= \frac{8e^4}{(p_1-p_3)^4} M E_1 E_3 \left[(E_1-E_3)(1-\cos\theta)+M(1+\cos\theta)\right] \\ &= \frac{8e^4}{(p_1-p_3)^4} 2M E_1 E_3 \left[(E_1-E_3)\sin^2(\theta/2)+M\cos^2(\theta/2)\right] \quad \textbf{(12)}\end{aligned}$$

• Now obtain expressions for $q^4=(p_1-p_3)^4$ and (E_1-E_3)

$$q^2=(p_1-p_3)^2=p_1^2+p_3^2-2p_1.p_3=-2E_1E_3(1-\cos\theta) \quad \textbf{(13)}$$

$$=-4E_1E_3\sin^2\theta/2 \quad \textbf{(14)}$$

NOTE: $q^2<0$ **Space-like**

• For (E_1-E_3) start from

$$q.p_2=(p_1-p_3).p_2=M(E_1-E_3)$$

and use

$$\begin{aligned}(q+p_2)^2 &= p_4^2 \\ q^2+p_2^2+2q.p_2 &= p_4^2 \\ q^2+M^2+2q.p_2 &= M^2 \\ \Rightarrow \quad q.p_2 &= -q^2/2\end{aligned}$$

$$q=(p_1-p_3)=(p_4-p_2)$$

•Hence the energy transferred to the proton:

$$E_1 - E_3 = -\frac{q^2}{2M} \tag{15}$$

Because q^2 is always negative $E_1 - E_3 > 0$ and the scattered electron is always lower in energy than the incoming electron

•Combining equations (11), (13) and (14):

$$\langle |M_{fi}|^2 \rangle = \frac{8e^4}{16E_1^2E_3^2\sin^4\theta/2} 2ME_1E_3 \left[M\cos^2\theta/2 - \frac{q^2}{2M}\sin^2\theta/2 \right]$$

$$= \frac{M^2e^4}{E_1E_3\sin^4\theta/2}\left[\cos^2\theta/2 - \frac{q^2}{2M^2}\sin^2\theta/2\right]$$

•For $E \gg m_e$ we have

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{E_3}{ME_1}\right)^2 |M_{fi}|^2$$

$$\alpha = \frac{e^2}{4\pi} \approx \frac{1}{137}$$

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2\sin^4\theta/2}\frac{E_3}{E_1}\left(\cos^2\theta/2 - \frac{q^2}{2M^2}\sin^2\theta/2\right) \tag{16}$$

Interpretation

- So far have derived the differential cross-section for e⁻p → e⁻p **elastic** scattering assuming point-like Dirac spin ½ particles. How should we interpret the equation?

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4\theta/2} \frac{E_3}{E_1}\left(\cos^2\theta/2 - \frac{q^2}{2M^2}\sin^2\theta/2\right)$$

- Compare with $$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} = \frac{\alpha^2}{4E^2\sin^4\theta/2}\cos^2\frac{\theta}{2}$$

the important thing to note about the Mott cross-section is that it is equivalent to scattering of spin ½ electrons in a fixed **electro-static** potential. Here the term E_3/E_1 is due to the proton recoil.

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4\theta/2} \frac{E_3}{E_1}\left(\cos^2\theta/2 - \underbrace{\frac{q^2}{2M^2}\sin^2\theta/2}\right)$$

- the new term: $\propto \sin^2\frac{\theta}{2}$ ⟷ **Magnetic interaction : due to the spin-spin interaction**

- The above differential cross-section depends on a single parameter. For an electron scattering angle θ, both q^2 and the energy, E_3, are fixed by kinematics

- Equating (13) and (15)

$$-2M(E_1 - E_3) = -2E_1E_3(1-\cos\theta)$$

$$\rightarrow \quad \frac{E_3}{E_1} = \frac{M}{M + E_1(1-\cos\theta)}$$

- Substituting back into (13):

$$\rightarrow \quad q^2 = -\frac{2ME_1^2(1-\cos\theta)}{M + E_1(1-\cos\theta)}$$

- e.g. e⁻p → e⁻p at E_{beam}= 529.5 MeV, look at scattered electrons at θ = 75°

For elastic scattering expect:

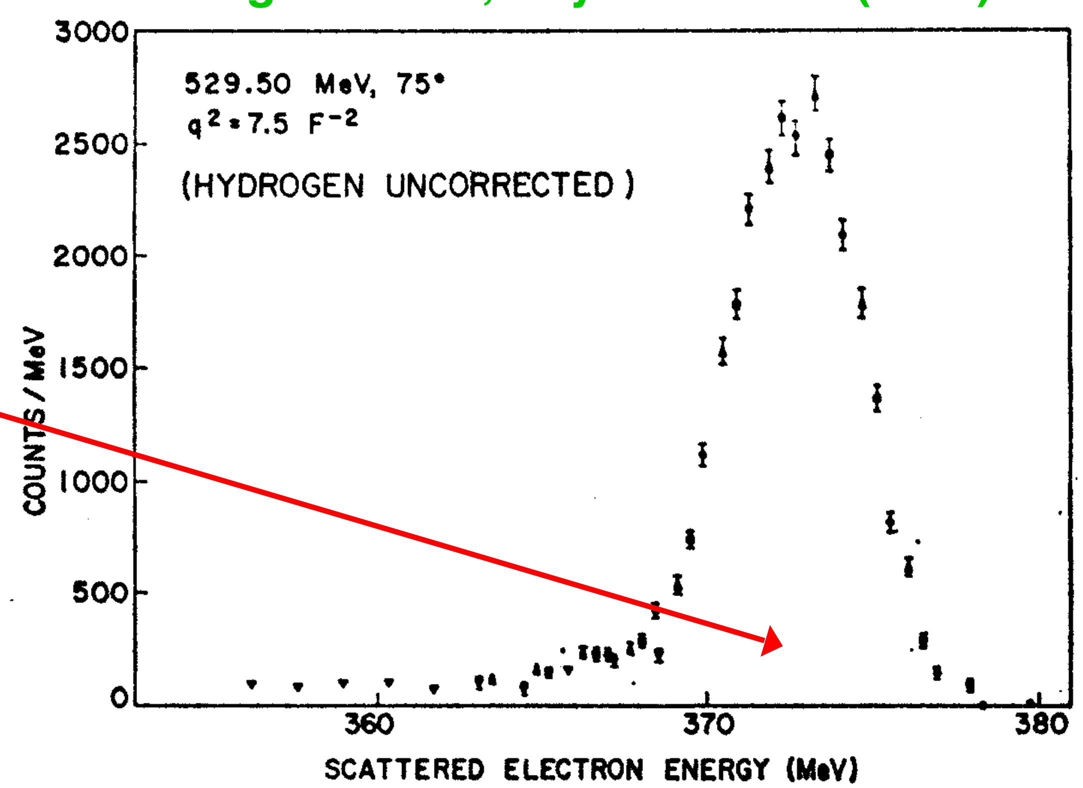
$$E_3 = \frac{ME_1}{M + E_1(1-\cos\theta)}$$

$$E_3 = \frac{938 \times 529}{938 + 529(1-\cos 75^\circ)} = 373 \text{ MeV}$$

The energy identifies the scatter as elastic.
Also know squared four-momentum transfer

$$|q^2| = \frac{2 \times 938 \times 529^2(1-\cos 75^\circ)}{938 + 529(1-\cos 75^\circ)} = 294 \text{MeV}^2$$

E.B.Hughes et al., Phys. Rev. 139 (1965) B458



Elastic Scattering from a Finite Size Proton

★**In general** the finite size of the proton can be accounted for by introducing two structure functions. One related to the charge distribution in the proton, $G_E(q^2)$ and the other related to the distribution of the magnetic moment of the proton, $G_M(q^2)$

- It can be shown that equation (16) generalizes to the **ROSENBLUTH FORMULA.**

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4 \theta/2} \frac{E_3}{E_1} \left(\frac{G_E^2 + \tau G_M^2}{(1+\tau)} \cos^2 \frac{\theta}{2} + 2\tau G_M^2 \sin^2 \frac{\theta}{2} \right)$$

with the Lorentz Invariant quantity: $\tau = -\frac{q^2}{4M^2} > 0$

- Unlike our previous discussion of form factors, here the form factors are a function of q^2 rather than $\vec{q}^2$ and cannot simply be considered in terms of the FT of the charge and magnetic moment distributions.

But $q^2 = (E_1 - E_3)^2 - \vec{q}^2$ and from eq (15) obtain

$$\Rightarrow \quad -\vec{q}^2 = q^2 \left[1 - \left(\frac{q}{2M} \right)^2 \right]$$

So for $\frac{q^2}{4M^2} \ll 1$ we have $q^2 \approx -\vec{q}^2$ and $G(q^2) \approx G(\vec{q}^2)$

- Hence in the limit $q^2/4M^2 \ll 1$ we can interpret the structure functions in terms of the Fourier transforms of the charge and magnetic moment distributions

$$G_E(q^2) \approx G_E(\vec{q}^2) = \int e^{i\vec{q}.\vec{r}} \rho(\vec{r}) \mathrm{d}^3\vec{r}$$

$$G_M(q^2) \approx G_M(\vec{q}^2) = \int e^{i\vec{q}.\vec{r}} \mu(\vec{r}) \mathrm{d}^3\vec{r}$$

- Note in deriving the Rosenbluth formula we assumed that the proton was a spin-half Dirac particle, i.e.

$$\vec{\mu} = \frac{e}{M}\vec{S}$$

- However, the experimentally measured value of the proton magnetic moment is larger than expected for a point-like Dirac particle:

$$\vec{\mu} = 2.79\frac{e}{M}\vec{S}$$

So for the proton expect

$$G_E(0) = \int \rho(\vec{r}) \mathrm{d}^3\vec{r} = 1 \qquad G_M(0) = \int \mu(\vec{r}) \mathrm{d}^3\vec{r} = \mu_p = +2.79$$

- Of course the anomalous magnetic moment of the proton is already evidence that it is not point-like !

Measuring $G_E(q^2)$ and $G_M(q^2)$

- Express the Rosenbluth formula as:

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_0 \left(\frac{G_E^2 + \tau G_M^2}{(1+\tau)} + 2\tau G_M^2 \tan^2\frac{\theta}{2}\right)$$

where $$\left(\frac{d\sigma}{d\Omega}\right)_0 = \frac{\alpha^2}{4E_1^2 \sin^4\theta/2}\frac{E_3}{E_1}\cos^2\frac{\theta}{2}$$

i.e. the Mott cross-section including the proton recoil. It corresponds to scattering from a spin-0 proton.

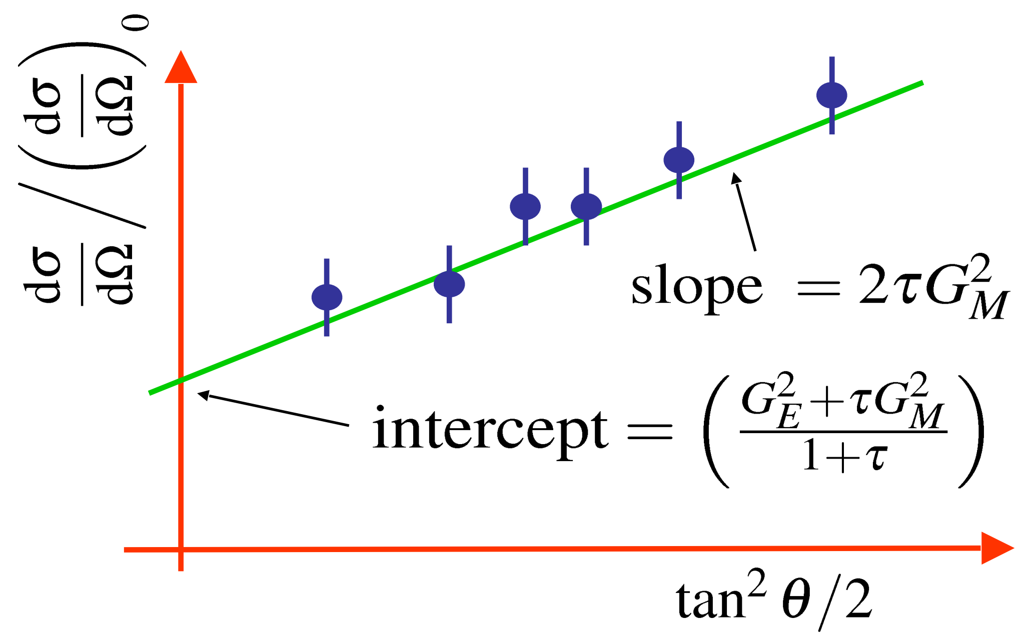
- At very low q^2: $\tau = -q^2/4M^2 \approx 0$

$$\frac{d\sigma}{d\Omega} \Big/ \left(\frac{d\sigma}{d\Omega}\right)_0 \approx G_E^2(q^2)$$

- At high q^2: $\tau \gg 1$

$$\frac{d\sigma}{d\Omega} \Big/ \left(\frac{d\sigma}{d\Omega}\right)_0 \approx \left(1 + 2\tau\tan^2\frac{\theta}{2}\right) G_M^2(q^2)$$

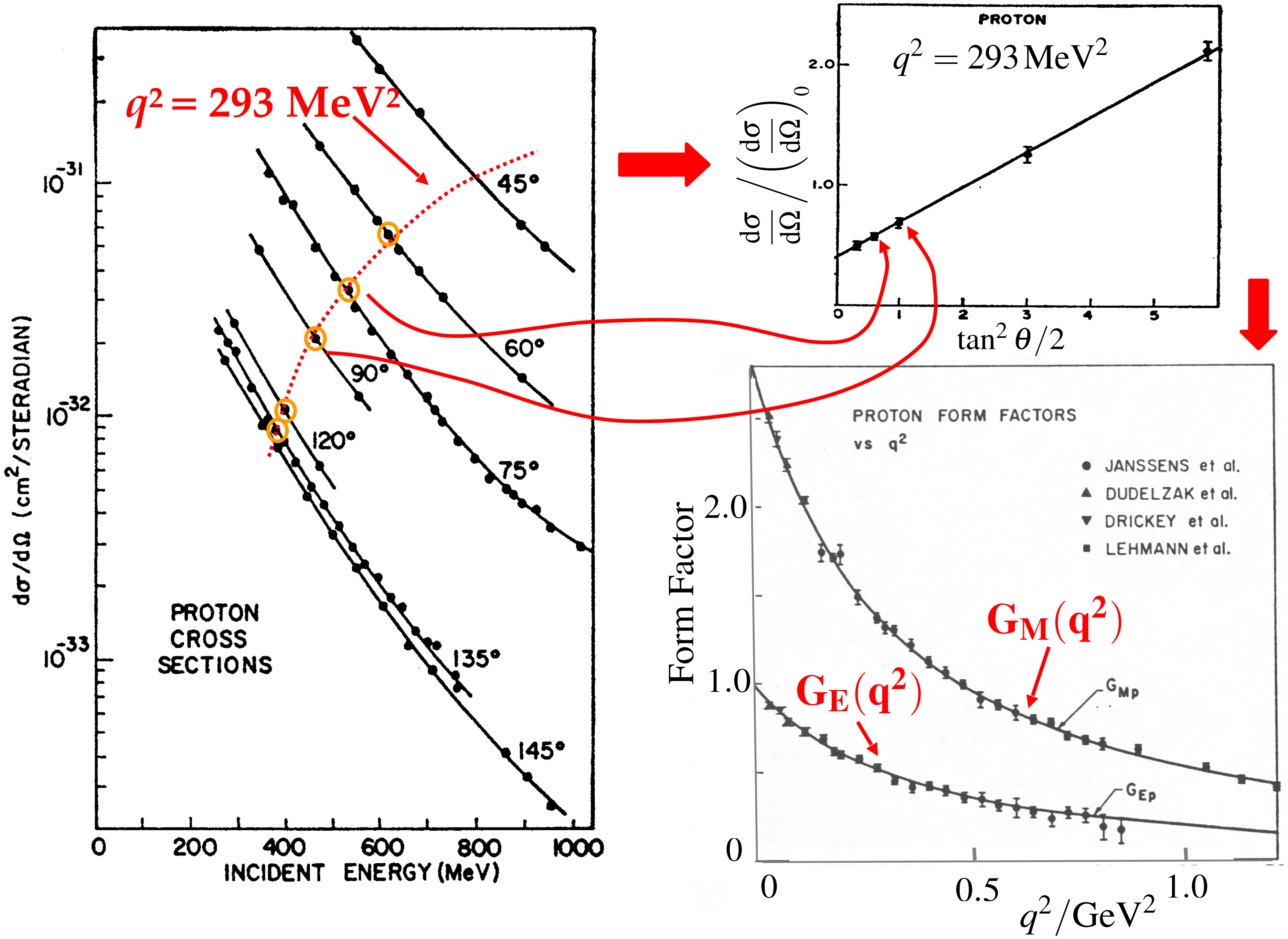
- In general we are sensitive to both structure functions! These can be resolved from the angular dependence of the cross section at **FIXED** q^2



EXAMPLE: e⁻p → e⁻p at E_{beam}= 529.5 MeV

- Electron beam energies chosen to give certain values of q^2
- Cross sections measured to 2-3 %

E.B.Hughes et al., Phys. Rev. 139 (1965) B458



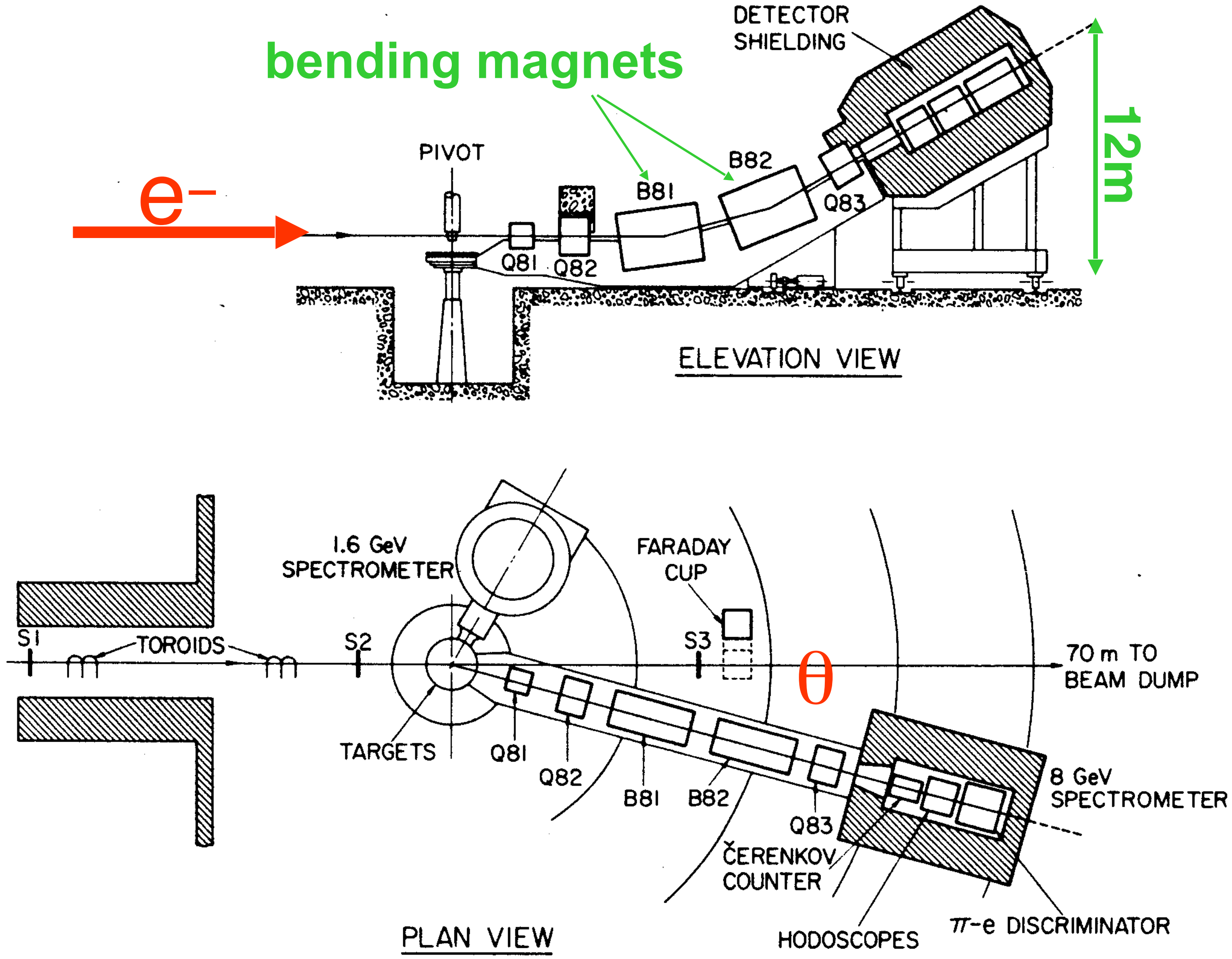
NOTE

Experimentally find $G_M(q^2) = 2.79 G_E(q^2)$, i.e. the electric and and magnetic form factors have same distribution

Higher Energy Electron-Proton Scattering

★Use electron beam from SLAC LINAC: **5 < E_{beam} < 20 GeV**

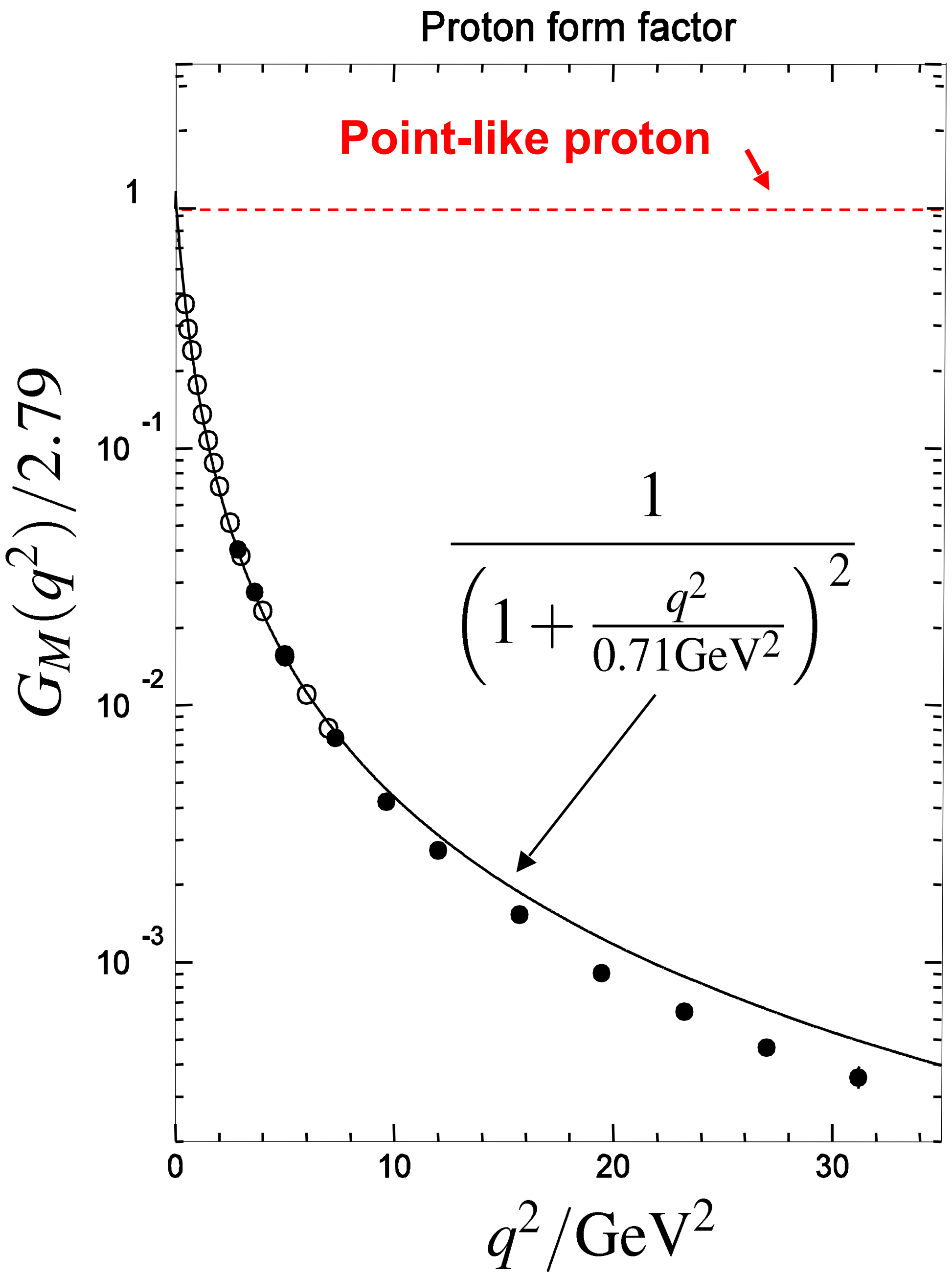
- Detect scattered electrons using the "8 GeV Spectrometer"



High q^2 → Measure $G_M(q^2)$

P.N.Kirk et al., Phys Rev D8 (1973) 63

High q^2 Results



R.C.Walker et al., Phys. Rev. D49 (1994) 5671
A.F.Sill et al., Phys. Rev. D48 (1993) 29

★ Form factor falls rapidly with q^2

- Proton is not point-like
- Good fit to the data with "dipole form":

$$G_E^p(q^2) \approx \frac{G_M^p}{2.79} \approx \frac{1}{(1+q^2/0.71\mathrm{GeV}^2)^2}$$

★ Taking FT find spatial charge and magnetic moment distribution

$$\rho(r) \approx \rho_0 e^{-r/a}$$

with $a \approx 0.24$ fm

- Corresponds to a rms charge radius

$$r_{rms} \approx 0.8 \text{ fm}$$

★ Although suggestive, does not imply proton is composite !

★ **Note:** so far have only considered ELASTIC scattering; Inelastic scattering is the subject of next handout

Summary: Elastic Scattering

★For elastic scattering of relativistic electrons from a point-like Dirac proton:

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4\theta/2} \frac{E_3}{E_1} \left(\cos^2\frac{\theta}{2} - \frac{q^2}{2M^2}\sin^2\frac{\theta}{2} \right)$$

Rutherford

Proton recoil

Electric/ Magnetic scattering

Magnetic term due to spin

★For elastic scattering of relativistic electrons from an extended proton:

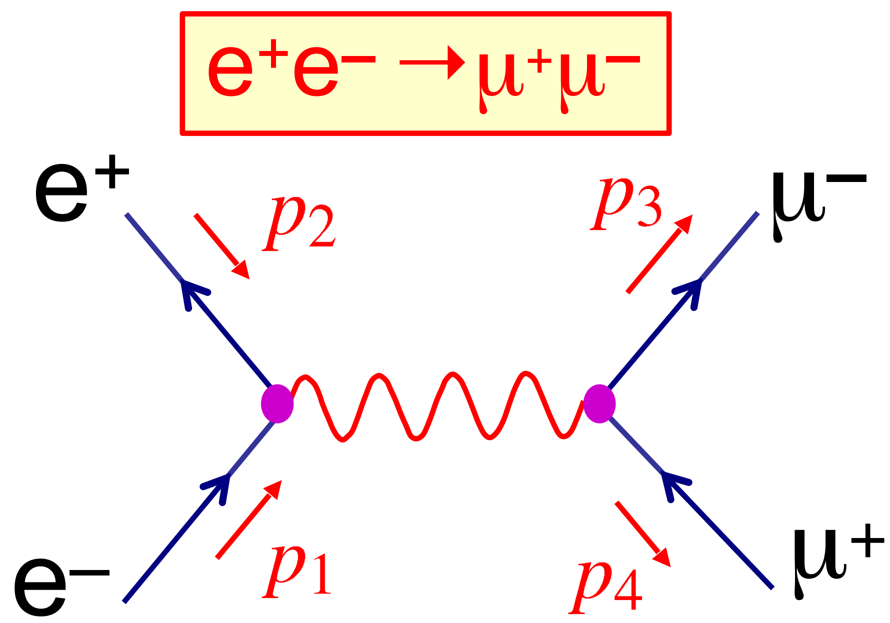
$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4\theta/2} \frac{E_3}{E_1} \left(\frac{G_E^2 + \tau G_M^2}{(1+\tau)} \cos^2\frac{\theta}{2} + 2\tau G_M^2 \sin^2\frac{\theta}{2} \right)$$

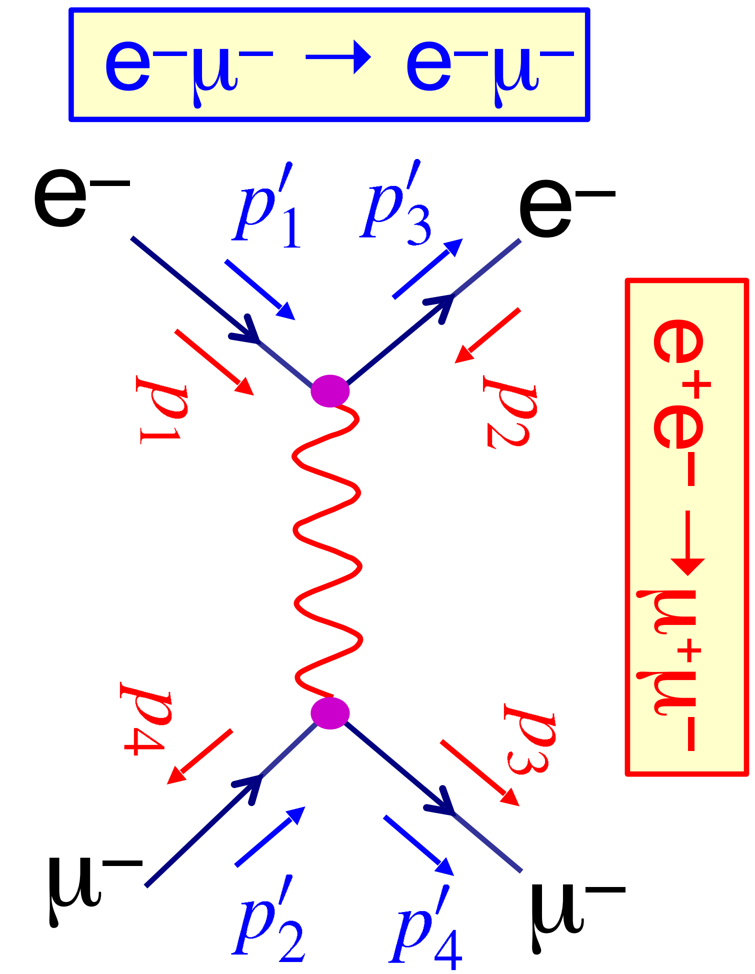
Rosenbluth Formula

★Electron elastic scattering from protons demonstrates that the proton is an extended object with rms charge radius of ~0.8 fm

Appendix I : Crossing Symmetry

★ **Having derived the Lorentz invariant matrix element for $e^+e^- \rightarrow \mu^+\mu^-$ "rotate" the diagram to correspond to $e^-\mu^- \rightarrow e^-\mu^-$ and apply the principle of crossing symmetry to write down the matrix element !**





★ **The transformation:**

$$p_1 \rightarrow p_1';\ p_2 \rightarrow -p_3';\ p_3 \rightarrow p_4';\ p_4 \rightarrow -p_2'$$

Changes the spin averaged matrix element for

$e^- e^+ \rightarrow \mu^- \mu^+$	→	$e^- \mu^- \rightarrow e^- \mu^-$
$p_1\ p_2\ \ \ p_3\ p_4$		$p_1'\ p_2'\ \ \ p_3'\ p_4'$