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Kern- und Teilchenphysik II
Lecture 1: QCD

(adapted from the Handout of Prof. Mark Thomson)

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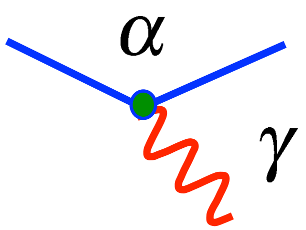
www.physik.uzh.ch/de/lehre/PHY213/FS2018.html

QCD

★The theory of the strong interaction, Quantum Chromodynamics (QCD), is very similar to QED but with 3 conserved "colour" charges

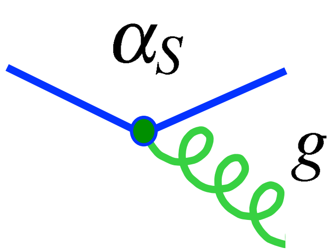
In QED:

- the electron carries one unit of charge $-e$
- the anti-electron carries one unit of anti-charge $+e$
- the force is mediated by a massless "gauge boson" – the photon



In QCD:

- quarks carry colour charge: r, g, b
- anti-quarks carry anti-charge: $\overline{r}, \overline{g}, \overline{b}$
- The force is mediated by massless gluons



★ In QCD, the strong interaction is invariant under rotations in colour space

$$r \leftrightarrow b;\ r \leftrightarrow g;\ b \leftrightarrow g$$

i.e. the same for all three colours

→ **SU(3) colour symmetry**

•This is an **exact** symmetry, unlike the approximate uds flavour symmetry discussed previously.

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Symmetries and Conservation Laws

★ **Suppose physics is invariant under the transformation**

$$\psi \rightarrow \psi' = \hat{U}\psi$$ **e.g. rotation of the coordinate axes**

• **To conserve probability normalisation require**

$$\langle\psi|\psi\rangle = \langle\psi'|\psi'\rangle = \langle\hat{U}\psi|\hat{U}\psi\rangle = \langle\psi|\hat{U}^\dagger\hat{U}|\psi\rangle$$

➡ $\boxed{\hat{U}^\dagger\hat{U} = 1}$ **i.e. $\hat{U}$ has to be unitary**

• **For physical predictions to be unchanged by the symmetry transformation, also require all QM matrix elements unchanged**

$$\langle\psi|\hat{H}|\psi\rangle = \langle\psi'|\hat{H}|\psi'\rangle = \langle\psi|\hat{U}^\dagger\hat{H}\hat{U}|\psi\rangle$$

i.e. require $\hat{U}^\dagger\hat{H}\hat{U} = \hat{H}$

$\times\hat{U}$ $\hat{U}\hat{U}^\dagger\hat{H}\hat{U} = \hat{U}\hat{H}$ ➡ $\hat{H}\hat{U} = \hat{U}\hat{H}$

therefore $\boxed{[\hat{H},\hat{U}] = 0}$ **$\hat{U}$ commutes with the Hamiltonian**

★ **Now consider the infinitesimal transformation (ε small)**

$$\hat{U} = 1 + i\varepsilon\hat{G}$$

($\hat{G}$ is called the generator of the transformation)

Symmetries and Conservation Laws

- For $\hat{U}$ to be unitary

$$\hat{U}\hat{U}^{\dagger} = (1+i\varepsilon\hat{G})(1-i\varepsilon\hat{G}^{\dagger}) = 1+i\varepsilon(\hat{G}-\hat{G}^{\dagger})+O(\varepsilon^2)$$

neglecting terms in ε^2 $\quad UU^{\dagger}=1$ ➡ $\boxed{\hat{G}=\hat{G}^{\dagger}}$

i.e. $\hat{G}$ is Hermitian and therefore corresponds to an observable quantity G !

- Furthermore, $[\hat{H},\hat{U}]=0 \quad \Rightarrow [\hat{H},1+i\varepsilon\,\hat{G}]=0 \quad \Rightarrow [\hat{H},\hat{G}]=0$

But from QM $\quad \dfrac{\mathrm{d}}{\mathrm{d}t}\langle\hat{G}\rangle = i\langle[\hat{H},\hat{G}]\rangle = 0$

i.e. G is a conserved quantity.

Symmetry ⟷ Conservation Law

★ **For each symmetry of nature have an observable conserved quantity**

Example: Infinitesimal spatial translation $x \rightarrow x+\varepsilon$

i.e. expect physics to be invariant under $\psi(x) \rightarrow \psi' = \psi(x+\varepsilon)$

$$\psi'(x) = \psi(x+\varepsilon) = \psi(x) + \frac{\partial\psi}{\partial x}\varepsilon = \left(1+\varepsilon\frac{\partial}{\partial x}\right)\psi(x)$$

but $\quad \hat{p}_x = -i\frac{\partial}{\partial x}$ ➡ $\psi'(x) = (1+i\varepsilon\hat{p}_x)\psi(x)$

The generator of the symmetry transformation is $\hat{p}_x$ ➡ p_x is conserved

- **Translational invariance** of physics implies **momentum conservation !**

Symmetries and Conservation Laws

- In general the symmetry operation may depend on more than one parameter

$$\hat{U} = 1 + i\vec{\varepsilon}.\vec{G}$$

For example for an infinitesimal 3D linear translation : $\vec{r} \rightarrow \vec{r} + \vec{\varepsilon}$

$$\rightarrow \quad \hat{U} = 1 + i\vec{\varepsilon}.\vec{p} \qquad \vec{p} = (\hat{p}_x, \hat{p}_y, \hat{p}_z)$$

- So far have only considered an infinitesimal transformation, however a finite transformation can be expressed as a series of infinitesimal transformations

$$\hat{U}(\vec{\alpha}) = \lim_{n\to\infty}\left(1 + i\frac{\vec{\alpha}}{n}.\vec{G}\right)^n = e^{i\vec{\alpha}.\vec{G}}$$

<u>**Example:**</u> Finite spatial translation in 1D: $x \rightarrow x + x_0$ with $\hat{U}(x_0) = e^{ix_0\hat{p}_x}$

$$\begin{aligned}\psi'(x) = \psi(x+x_0) &= \hat{U}\psi(x) = \exp\left(x_0\frac{\mathrm{d}}{\mathrm{d}x}\right)\psi(x) \qquad \left(p_x = -i\frac{\partial}{\partial x}\right)\\ &= \left(1 + x_0\frac{\mathrm{d}}{\mathrm{d}x} + \frac{x_0^2}{2!}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \ldots\right)\psi(x)\\ &= \psi(x) + x_0\frac{\mathrm{d}\psi}{\mathrm{d}x} + \frac{x_0^2}{2}\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + \ldots\end{aligned}$$

i.e. obtain the expected Taylor expansion

Isospin

- The proton and neutron have very similar masses and the nuclear force is found to be approximately charge-independent, i.e.

$$V_{pp} \approx V_{np} \approx V_{nn}$$

- To reflect this symmetry, Heisenberg (1932) proposed that if you could "switch off" the electric charge of the proton

There would be no way to distinguish between a proton and neutron

- Proposed that the neutron and proton should be considered as two states of a single entity; the **nucleon**

$$p = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad n = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

★ Analogous to the spin-up/spin-down states of a spin-½ particle

ISOSPIN

★ Expect physics to be invariant under rotations in this space

- The neutron and proton form an isospin doublet with total isospin $I = 1/2$ and third component $I_3 = \pm 1/2$

Flavour Symmetry

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We can extend this idea to the quarks:

★ **Assume the strong interaction treats all quark flavours equally (it does)**

- **Because $m_u \approx m_d$:**

  The strong interaction possesses an approximate flavour symmetry
  i.e. from the point of view of the strong interaction nothing changes if all up quarks are replaced by down quarks and *vice versa*.

- **Choose the basis**

$$u = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad d = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- **Express the invariance of the strong interaction under** $u \leftrightarrow d$ **as invariance under "rotations" in an abstract isospin space**

$$\begin{pmatrix} u' \\ d' \end{pmatrix} = \hat{U} \begin{pmatrix} u \\ d \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix}$$

The 2x2 unitary matrix depends on 4 complex numbers, i.e. 8 real parameters
But there are four constraints from $\hat{U}^\dagger \hat{U} = 1$

➡ **8 – 4 = 4 independent matrices**

- **In the language of group theory the four matrices form the U(2) group**

Flavour Symmetry

- One of the matrices corresponds to multiplying by a phase factor

$$\hat{U}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} e^{i\phi}$$

not a flavour transformation and of no relevance here.

- The remaining three matrices form an SU(2) group (special unitary) with $\det U = 1$
- For an infinitesimal transformation, in terms of the Hermitian generators $\hat{G}$

$$\hat{U} = 1 + i\varepsilon\hat{G}$$

- $\det U = 1 \quad \Rightarrow \quad Tr(\hat{G}) = 0$
- A linearly independent choice for $\hat{G}$ are the Pauli spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- The proposed flavour symmetry of the strong interaction has the same transformation properties as SPIN !
- Define ISOSPIN: $\vec{T} = \frac{1}{2}\vec{\sigma} \qquad \hat{U} = e^{i\vec{\alpha}.\vec{T}}$
- Check this works, for an infinitesimal transformation

$$\hat{U} = 1 + \frac{1}{2}i\vec{\varepsilon}.\vec{\sigma} \quad = \quad 1 + \frac{i}{2}(\varepsilon_1\sigma_1 + \varepsilon_2\sigma_2 + \varepsilon_3\sigma_3) = \begin{pmatrix} 1 + \frac{1}{2}i\varepsilon_3 & \frac{1}{2}i(\varepsilon_1 - i\varepsilon_2) \\ \frac{1}{2}i(\varepsilon_1 + i\varepsilon_2) & 1 - \frac{1}{2}i\varepsilon_3 \end{pmatrix}$$

Which is, as required, unitary and has unit determinant

$$U^\dagger U = I + O(\varepsilon^2) \qquad \det U = 1 + O(\varepsilon^2)$$

Properties of Isospin

- **Isospin has the exactly the same properties as spin**

$$[T_1, T_2] = iT_3 \quad [T_2, T_3] = iT_1 \quad [T_3, T_1] = iT_2$$

$$[T^2, T_3] = 0 \qquad T^2 = T_1^2 + T_2^2 + T_3^2$$

As in the case of spin, have three non-commuting operators, T_1, T_2, T_3 and even though all three correspond to observables, can't know them simultaneously. So label states in terms of total isospin I and the third component of isospin I_3

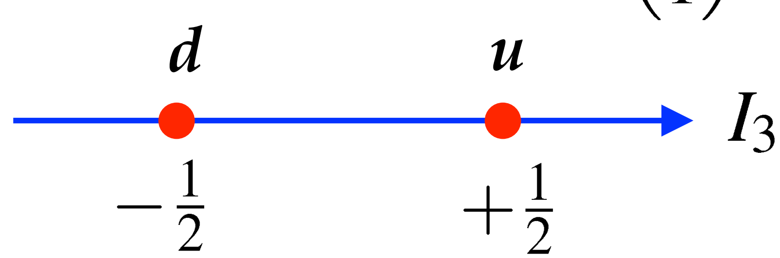
NOTE: isospin has nothing to do with spin – just the same mathematics

- **The eigenstates are exact analogues of the eigenstates of ordinary angular momentum** $|s, m\rangle \rightarrow |I, I_3\rangle$

with $T^2|I, I_3\rangle = I(I+1)|I, I_3\rangle \qquad T_3|I, I_3\rangle = I_3|I, I_3\rangle$

- **In terms of isospin:**

$$u = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |\ \tfrac{1}{2}, +\tfrac{1}{2}\ \rangle \qquad d = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |\tfrac{1}{2}, -\tfrac{1}{2}\rangle$$



$$I = \tfrac{1}{2}, \quad I_3 = \pm\tfrac{1}{2}$$

- **In general** $I_3 = \frac{1}{2}(N_u - N_d)$

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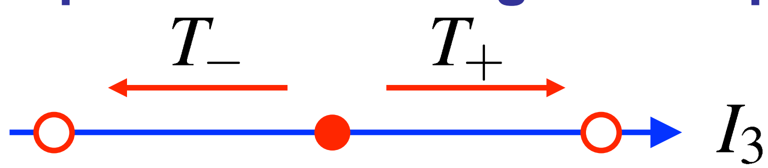
Properties of Isospin

- Can define isospin ladder operators – analogous to spin ladder operators

$T_- \equiv T_1 - iT_2$

u → d



$T_+ \equiv T_1 + iT_2$

d → u

$$T_+|I,I_3\rangle = \sqrt{I(I+1)-I_3(I_3+1)}\,|I,I_3+1\rangle$$

$$T_-|I,I_3\rangle = \sqrt{I(I+1)-I_3(I_3-1)}\,|I,I_3-1\rangle$$

Step up/down in I_3 until reach end of multiplet $T_+|I,+I\rangle = 0 \quad T_-|I,-I\rangle = 0$

$$T_+u = 0 \qquad T_+d = u \qquad T_-u = d \qquad T_-d = 0$$

- Ladder operators turn $u \to d$ and $d \to u$

★ Combination of isospin: e.g. what is the isospin of a system of two **d** quarks, is exactly analogous to combination of spin (i.e. angular momentum)

$$|I^{(1)},I_3^{(1)}\rangle|I^{(2)},I_3^{(2)}\rangle \to |I,I_3\rangle$$

- I_3 additive : $I_3 = I_3^{(1)} + I_3^{(2)}$
- I in integer steps from $|I^{(1)} - I^{(2)}|$ to $|I^{(1)} + I^{(2)}|$

★ Assumed symmetry of Strong Interaction under isospin transformations implies the existence of conserved quantites

- In strong interactions I_3 and I are conserved, analogous to conservation of J_z and J for angular momentum

SU(3) Flavour

★ Extend these ideas to include the strange quark. Since $m_s > m_u, m_d$ don't have an exact symmetry. But m_s not so very different from m_u, m_d and can treat the strong interaction (and resulting hadron states) as if it were symmetric under $u \leftrightarrow d \leftrightarrow s$

- **NOTE:** any results obtained from this assumption are only approximate as the symmetry is not exact.
- The assumed uds flavour symmetry can be expressed as

$$\begin{pmatrix} u' \\ d' \\ s' \end{pmatrix} = \hat{U} \begin{pmatrix} u \\ d \\ s \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{pmatrix} \begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

- The 3x3 unitary matrix depends on 9 complex numbers, i.e. 18 real parameters There are 9 constraints from $\hat{U}^\dagger \hat{U} = 1$

➡ Can form 18 – 9 = 9 linearly independent matrices

These 9 matrices form a U(3) group.

- As before, one matrix is simply the identity multiplied by a complex phase and is of no interest in the context of flavour symmetry
- The remaining 8 matrices have $\det U = 1$ and form an SU(3) group
- The eight matrices (the Hermitian generators) are: $\vec{T} = \frac{1}{2}\vec{\lambda}$ $\quad \hat{U} = e^{i\vec{\alpha}.\vec{T}}$

SU(3) Flavour

★ In SU(3) flavour, the three quark states are represented by:

$$u = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad d = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad s = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

★ In SU(3) **uds** flavour symmetry contains SU(2) **ud** flavour symmetry which allows us to write the first three matrices:

$$\lambda_1 = \begin{pmatrix} \sigma_1 & & 0 \\ & & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} \sigma_2 & & 0 \\ & & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} \sigma_3 & & 0 \\ & & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

i.e. **u ↔ d**

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- The third component of isospin is now written $I_3 = \frac{1}{2}\lambda_3$

  with $I_3 u = +\frac{1}{2}u \quad I_3 d = -\frac{1}{2}d \quad I_3 s = 0$

- I_3 "counts the number of up quarks – number of down quarks in a state
- As before, ladder operators $T_\pm = \frac{1}{2}(\lambda_1 \pm i\lambda_2)$

$$d \bullet \longleftarrow T_\pm \longrightarrow \bullet\, u$$

SU(3) Flavour

- **Now consider the matrices corresponding to the u ↔ s and d ↔ s**

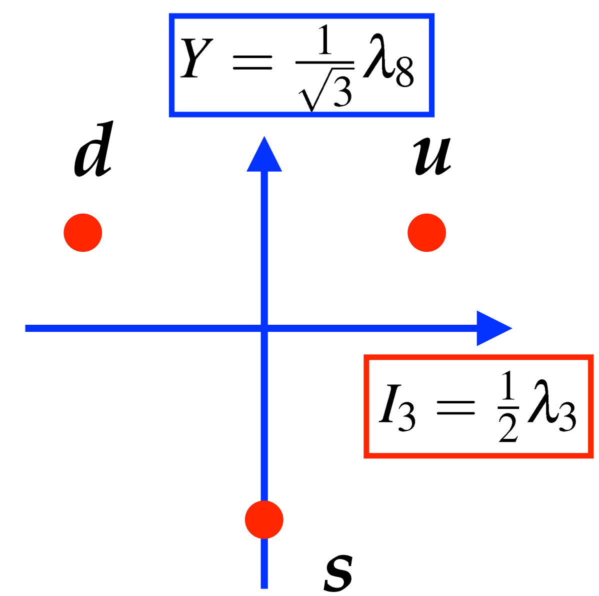
u ↔ s $\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ $\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

d ↔ s $\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ $\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

- **Hence in addition to** $\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ **have two other traceless diagonal matrices**
- **However the three diagonal matrices are not be independent.**
- **Define the eighth matrix, λ_8, as the linear combination:**

$$\lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} + \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

which specifies the "vertical position" in the 2D plane

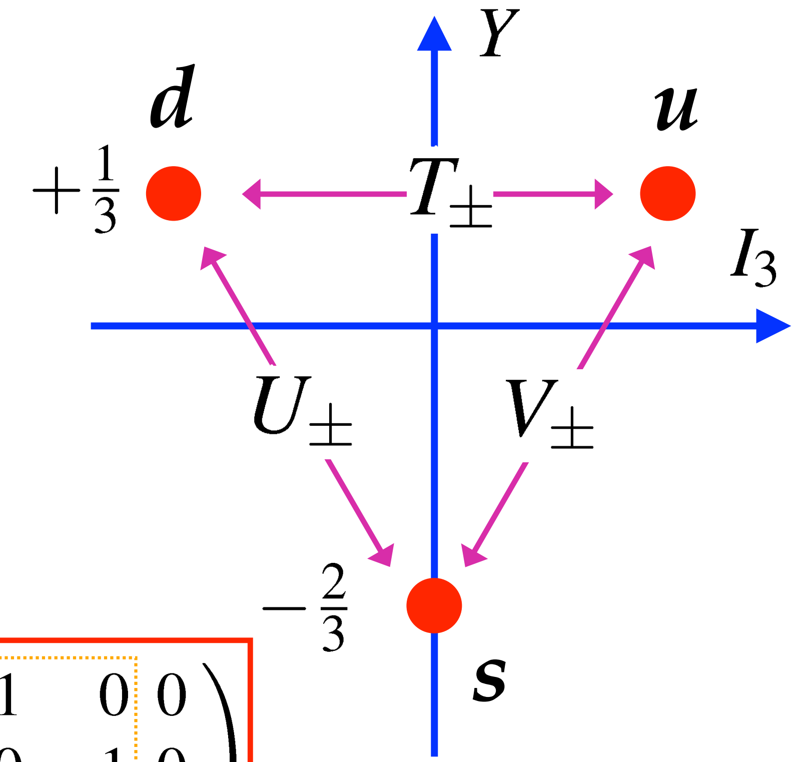


"Only need two axes (quantum numbers) to specify a state in the 2D plane": (I_3,Y)

SU(3) Flavour

★ **The other six matrices form six ladder operators which step between the states**

$$T_\pm = \tfrac{1}{2}(\lambda_1 \pm i\lambda_2)$$
$$V_\pm = \tfrac{1}{2}(\lambda_4 \pm i\lambda_5)$$
$$U_\pm = \tfrac{1}{2}(\lambda_6 \pm i\lambda_7)$$



with $I_3 = \frac{1}{2}\lambda_3$ $\quad Y = \frac{1}{\sqrt{3}}\lambda_8$

and the eight Gell-Mann matrices

u ↔ d

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

u ↔ s

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$

d ↔ s

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$\lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

QCD

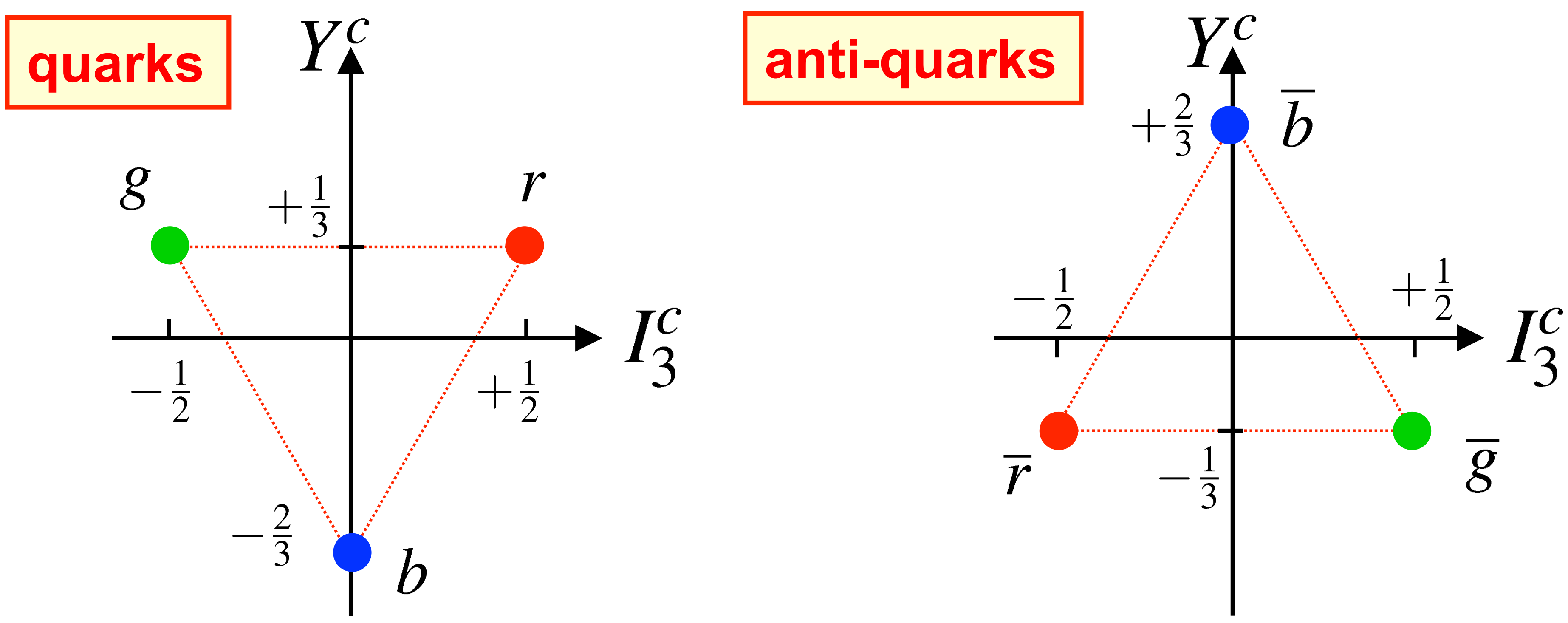
★ Represent r, g, b SU(3) colour states by:

$$r = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \quad g = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; \quad b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

★ Colour states can be labelled by two quantum numbers:
- I_3^c colour isospin
- Y^c colour hypercharge

Exactly analogous to labelling u,d,s flavour states by I_3 and Y

★ Each quark (anti-quark) can have the following colour quantum numbers:



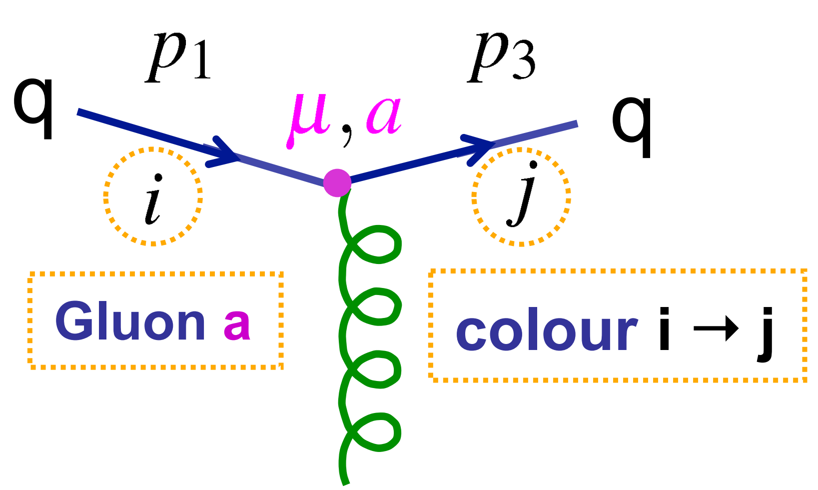
Quark-gluon interaction

- Representing the colour part of the fermion wave-functions by:

$$r = c_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad g = c_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad b = c_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

- Particle wave-functions $u(p) \longrightarrow c_i u(p)$
- The QCD qqg vertex is written:

$$\overline{u}(p_3)c_j^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_i u(p_1)$$



- Only difference w.r.t. QED is the insertion of the 3x3 SU(3) Gell-Mann matrices
- Isolating the colour part:

$$c_j^\dagger \lambda^a c_i = c_j^\dagger \begin{pmatrix} \lambda^a_{1i} \\ \lambda^a_{2i} \\ \lambda^a_{3i} \end{pmatrix} = \lambda^a_{ji}$$

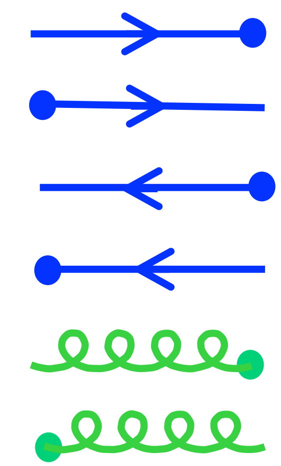
- Hence the fundamental quark - gluon QCD interaction can be written

$$\overline{u}(p_3)c_j^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_i u(p_1) \equiv \overline{u}(p_3)\{-\tfrac{1}{2}ig_s\lambda^a_{ji}\gamma^\mu\}u(p_1)$$

Feynman Rules for QCD

- **External Lines**

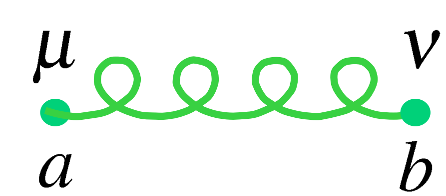
spin 1/2	incoming quark	$u(p)$	
	outgoing quark	$\overline{u}(p)$	
	incoming anti-quark	$\overline{v}(p)$	
	outgoing anti-quark	$v(p)$	
spin 1	incoming gluon	$\varepsilon^{\mu}(p)$	
	outgoing gluon	$\varepsilon^{\mu}(p)^{*}$	



- **Internal Lines (propagators)**

spin 1 gluon

$$\boxed{\frac{-ig_{\mu\nu}}{q^2}\delta^{ab}}$$

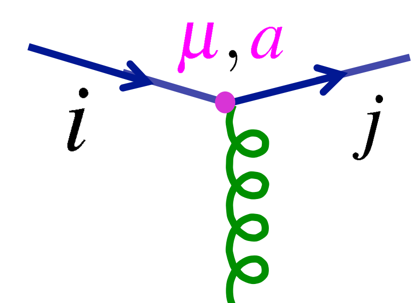


a, b = 1,2,…,8 are gluon colour indices

- **Vertex Factors**

spin 1/2 quark

$$\boxed{-ig_s\frac{1}{2}\lambda^{a}_{ji}\gamma^{\mu}}$$



i, j = 1,2,3 are quark colours,

λ^a a = 1,2,..8 are the Gell-Mann SU(3) matrices

- **+ 3 gluon and 4 gluon interaction vertices**
- **Matrix Element** $-iM$ = **product of all factors**

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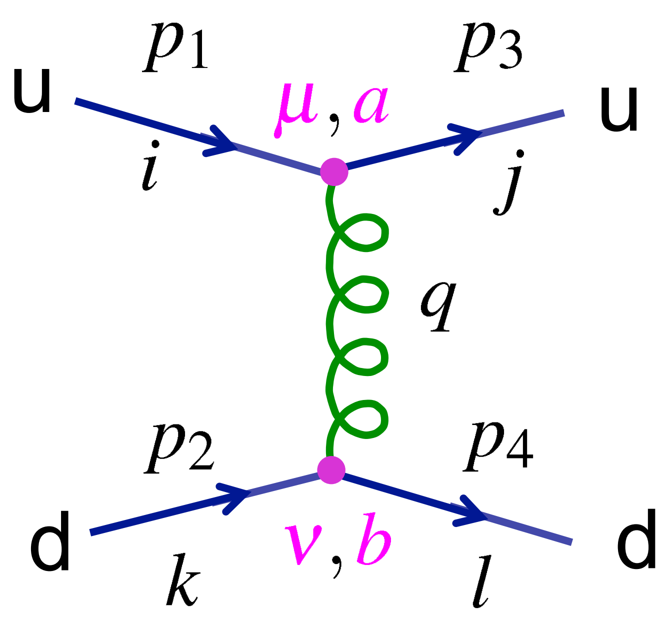
QCD Scattering

★ **Consider QCD scattering of an up and a down quark**



- **The incoming and out-going quark colours are labelled by** $i, j, k, l = \{1,2,3\}$ (or $\{r, g, b\}$)
- **In terms of colour this scattering is**
$$ik \rightarrow jl$$
- **The 8 different gluons are accounted for by the colour indices** $a, b = 1, 2, \ldots, 8$
- **NOTE: the δ-function in the propagator ensures** $a = b$**, i.e. the gluon "emitted" at** a **is the same as that "absorbed" at** b

★ **Applying the Feynman rules:**

$$-iM = \left[\bar{u}_u(p_3)\{-\tfrac{1}{2}ig_s\lambda^a_{ji}\gamma^\mu\}u_u(p_1)\right]\frac{-ig_{\mu\nu}}{q^2}\delta^{ab}\left[\bar{u}_d(p_4)\{-\tfrac{1}{2}ig_s\lambda^b_{lk}\gamma^\nu\}u_d(p_2)\right]$$

where summation over a **and** b **(and** μ **and** ν**) is implied.**

★ **Summing over** a **and** b **using the δ-function gives:**

$$M = -\frac{g_s^2}{4}\lambda^a_{ji}\lambda^a_{lk}\frac{1}{q^2}g_{\mu\nu}[\bar{u}_u(p_3)\gamma^\mu u_u(p_1)][\bar{u}_d(p_4)\gamma^\nu u_d(p_2)]$$

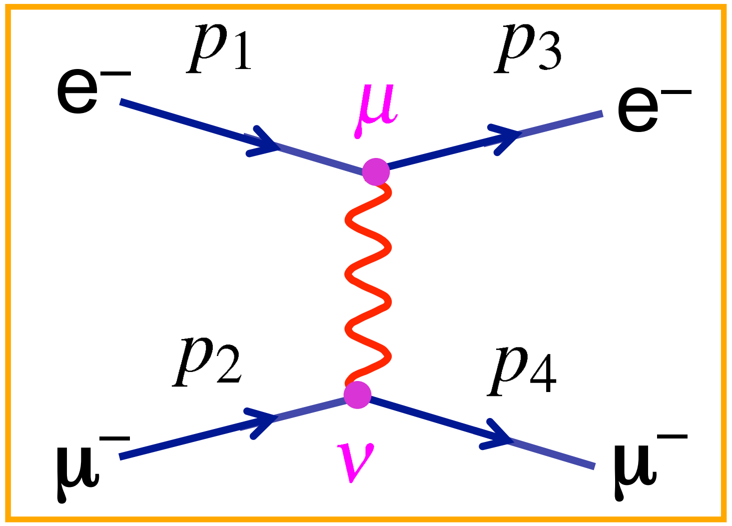
Sum over all 8 gluons (repeated indices)

QED vs QCD

QED

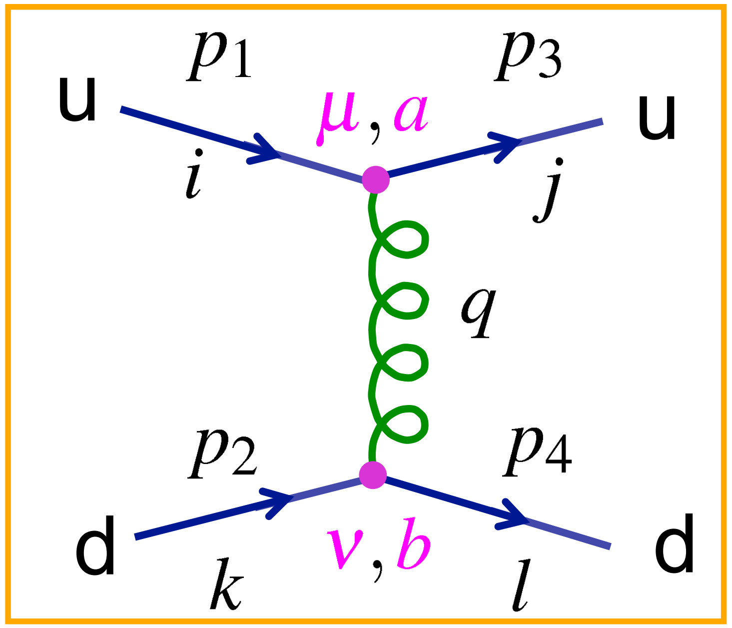
$$-iM = [\overline{u}(p_3)ie\gamma^\mu u(p_1)]\frac{-ig_{\mu\nu}}{q^2}[\overline{u}(p_4)ie\gamma^\nu u(p_2)]$$

$$M = -e^2\frac{1}{q^2}g_{\mu\nu}[\overline{u}(p_3)\gamma^\mu u(p_1)][\overline{u}(p_4)\gamma^\nu u(p_2)]$$



QCD

$$M = -\frac{g_s^2}{4}\lambda^a_{ji}\lambda^a_{lk}\frac{1}{q^2}g_{\mu\nu}[\overline{u}_u(p_3)\gamma^\mu u_u(p_1)][\overline{u}_d(p_4)\gamma^\nu u_d(p_2)]$$



★ **QCD Matrix Element = QED Matrix Element with:**

- $e^2 \rightarrow g_s^2$ **or equivalently** $\alpha = \frac{e^2}{4\pi} \rightarrow \alpha_s = \frac{g_s^2}{4\pi}$

+ QCD Matrix Element includes an additional "colour factor"

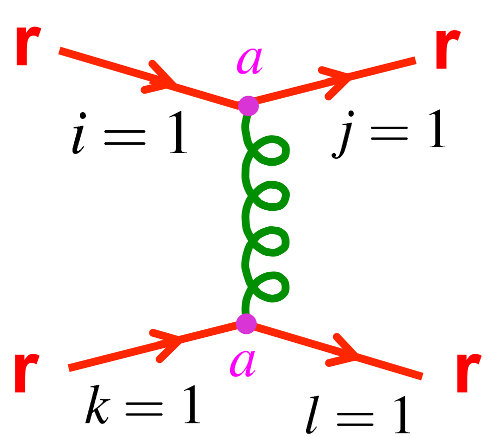
$$C(ik \rightarrow jl) \equiv \frac{1}{4}\sum_{a=1}^{8}\lambda^a_{ji}\lambda^a_{lk}$$

QCD Color Factor

- **QCD colour factors reflect the gluon states that are involved**

	$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	$\lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
	$\lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$	$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$	$\lambda^8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$
Gluons:	$r\overline{g}\,,\, g\overline{r}$	$r\overline{b}\,,\, b\overline{r}$	$g\overline{b}\,,\, b\overline{g}$	$\frac{1}{\sqrt{2}}(r\overline{r}-g\overline{g})$ $\quad \frac{1}{\sqrt{6}}(r\overline{r}+g\overline{g}-2b\overline{b})$

❶ Configurations involving a single colour



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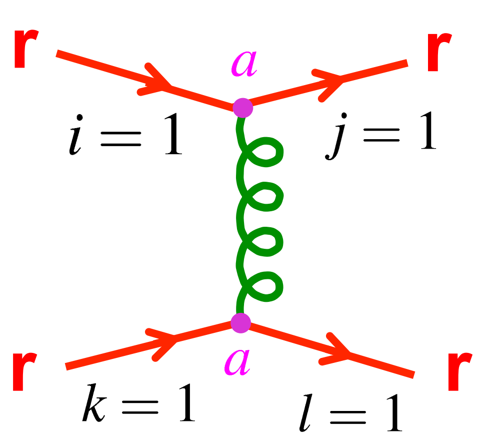
QCD Color Factor

- QCD colour factors reflect the gluon states that are involved

$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	$\lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
$\lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$	$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$	$\lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$
Gluons: $r\bar{g}, g\bar{r}$	$r\bar{b}, b\bar{r}$	$g\bar{b}, b\bar{g}$	$\frac{1}{\sqrt{2}}(r\bar{r} - g\bar{g})$ $\frac{1}{\sqrt{6}}(r\bar{r} + g\bar{g} - 2b\bar{b})$

❶ Configurations involving a single colour



- Only matrices with non-zero entries in **11** position are involved

$$C(rr \to rr) = \frac{1}{4}\sum_{a=1}^{8} \lambda^a_{11}\lambda^a_{11} = \frac{1}{4}(\lambda^3_{11}\lambda^3_{11} + \lambda^8_{11}\lambda^8_{11})$$

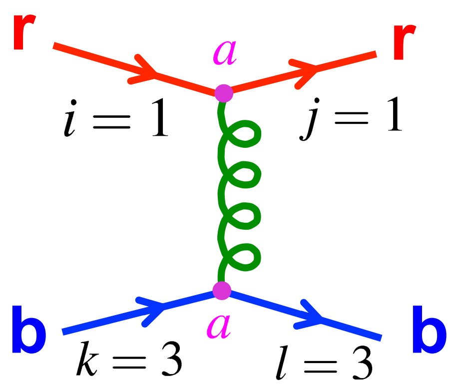
$$= \frac{1}{4}\left(1 + \frac{1}{3}\right) = \frac{1}{3}$$

Similarly find

$$C(rr \to rr) = C(gg \to gg) = C(bb \to bb) = \frac{1}{3}$$

QCD Color Factor

❷ Other configurations where quarks don't change colour e.g. $rb \to rb$



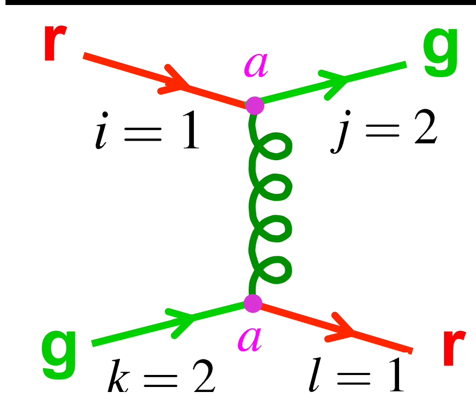
- Only matrices with non-zero entries in 11 and 33 position are involved

$$C(rb\to rb) = \frac{1}{4}\sum_{a=1}^{8}\lambda^a_{11}\lambda^a_{33} = \frac{1}{4}(\lambda^8_{11}\lambda^8_{33})$$

$$= \frac{1}{4}\left(\frac{1}{\sqrt{3}}\cdot\frac{-2}{\sqrt{3}}\right) = -\frac{1}{6}$$

Similarly $C(rb\to rb) = C(rg\to rg) = C(gr\to gr) = C(gb\to gb) = C(br\to br) = C(bg\to bg) = -\frac{1}{6}$

❸ Configurations where quarks swap colours e.g. $rg \to gr$



- Only matrices with non-zero entries in 12 and 21 position are involved

Gluons $r\bar{g}\,,\, g\bar{r}$

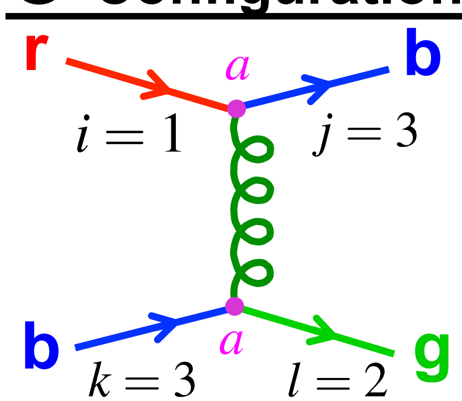
$$C(rg\to gr) = \frac{1}{4}\sum_{a=1}^{8}\lambda^a_{21}\lambda^a_{12} = \frac{1}{4}(\lambda^1_{21}\lambda^1_{12} + \lambda^2_{21}\lambda^2_{12})$$

$$= \frac{1}{4}(i(-i)+1) = \frac{1}{2}$$

$\hat{T}^{(ij)}_{+}\hat{T}^{(kl)}_{-}$

$$C(rb\to br) = C(rg\to gr) = C(gr\to rg) = C(gb\to bg) = C(br\to rb) = C(bg\to gb) = \frac{1}{2}$$

❹ Configurations involving 3 colours e.g. $rb \to bg$



- Only matrices with non-zero entries in the 13 and 32 position
- But none of the λ matrices have non-zero entries in the 13 and 32 positions. Hence the colour factor is zero

★ colour is conserved

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QCD Color Factor (anti-quark)

- **Recall the colour part of wave-function:**

$$r = c_1 = \begin{pmatrix}1\\0\\0\end{pmatrix} \quad g = c_2 = \begin{pmatrix}0\\1\\0\end{pmatrix} \quad b = c_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

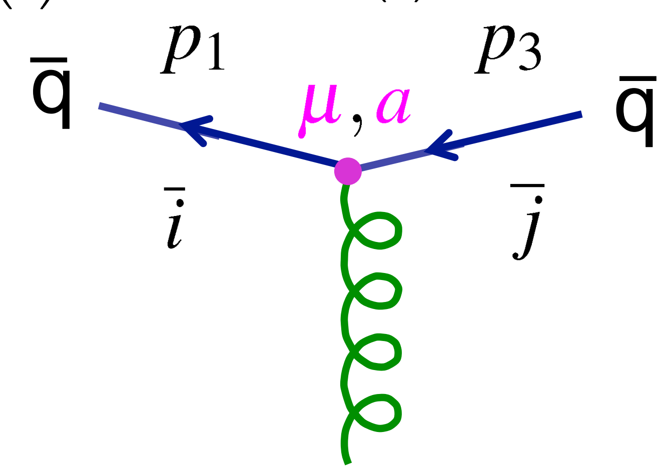
- **The QCD qqg vertex was written:**

$$\overline{u}(p_3)c_j^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_i u(p_1)$$

★ **Now consider the anti-quark vertex**

- **The QCD $\bar{q}\bar{q}$g vertex is:**

$$\overline{v}(p_1)c_i^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_j v(p_3)$$



Note that the incoming anti-particle now enters on the LHS of the expression

- **For which the colour part is**

$$c_i^\dagger\lambda^a c_j = c_i^\dagger \begin{pmatrix}\lambda^a_{1j}\\ \lambda^a_{2j}\\ \lambda^a_{3j}\end{pmatrix} = \lambda^a_{ij}$$

i.e indices *ij* are swapped with respect to the quark case

- **Hence**

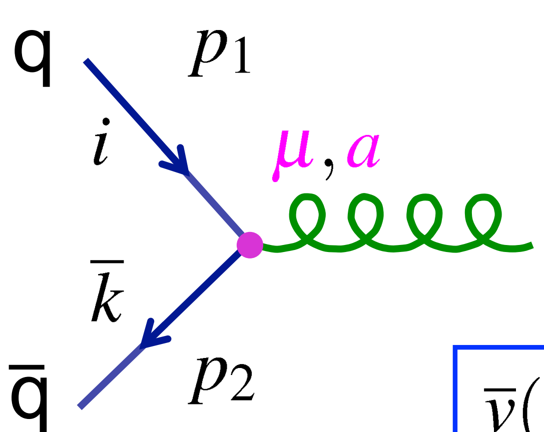
$$\overline{v}(p_1)c_i^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_j v(p_3) \equiv \overline{v}(p_1)\{-\tfrac{1}{2}ig_s\lambda^a_{ij}\gamma^\mu\}v(p_3)$$

- **c.f. the quark - gluon QCD interaction**

$$\overline{u}(p_3)c_j^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_i u(p_1) \equiv \overline{u}(p_3)\{-\tfrac{1}{2}ig_s\lambda^a_{ji}\gamma^\mu\}u(p_1)$$

Quark-Antiquark annihilation

★ **Finally we can consider the quark – anti-quark annihilation**



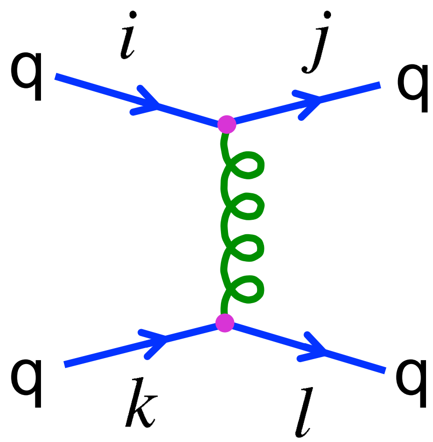
QCD vertex: $\overline{v}(p_2)c_k^\dagger\{-\frac{1}{2}ig_s\lambda^a\gamma^\mu\}c_i u(p_1)$

with $c_k^\dagger \lambda^a c_i = \lambda^a_{ki}$

$$\overline{v}(p_2)c_k^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_i u(p_1) \equiv \overline{v}(p_2)\{-\tfrac{1}{2}ig_s\lambda^a_{ki}\gamma^\mu\}u(p_1)$$

QCD Color Factor

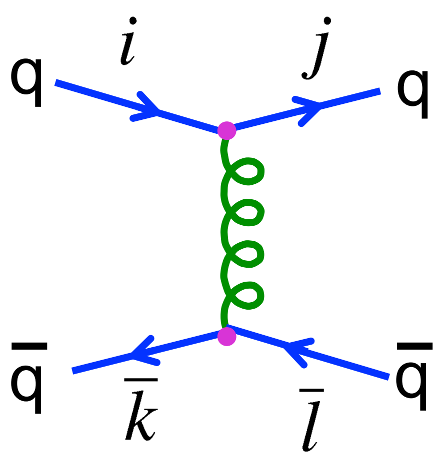
- Consequently the colour factors for the different diagrams are:



$$C(ik \to jl) \equiv \frac{1}{4}\sum_{a=1}^{8} \lambda^a_{ji}\lambda^a_{lk}$$

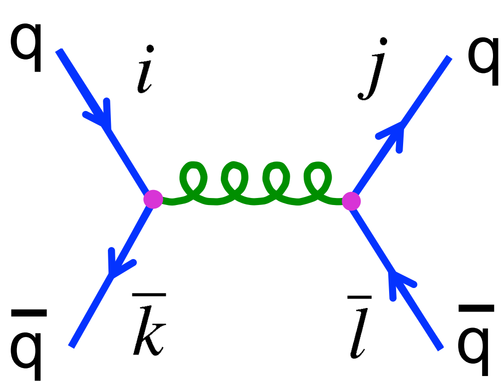
e.g.

$$C(rr \to rr) = \tfrac{1}{3}$$
$$C(rg \to rg) = -\tfrac{1}{6}$$
$$C(rg \to gr) = \tfrac{1}{2}$$



$$C(i\bar{k} \to j\bar{l}) \equiv \frac{1}{4}\sum_{a=1}^{8} \lambda^a_{ji}\lambda^a_{kl}$$

$$C(r\bar{r} \to r\bar{r}) = \tfrac{1}{3}$$
$$C(r\bar{g} \to r\bar{g}) = -\tfrac{1}{6}$$
$$C(r\bar{r} \to g\bar{g}) = \tfrac{1}{2}$$



$$C(i\bar{k} \to j\bar{l}) \equiv \frac{1}{4}\sum_{a=1}^{8} \lambda^a_{ki}\lambda^a_{jl}$$

$$C(r\bar{r} \to r\bar{r}) = \tfrac{1}{3}$$
$$C(r\bar{g} \to r\bar{g}) = \tfrac{1}{2}$$
$$C(r\bar{r} \to g\bar{g}) = -\tfrac{1}{6}$$

Colour index of adjoint spinor comes first

Quark-antiquark scattering

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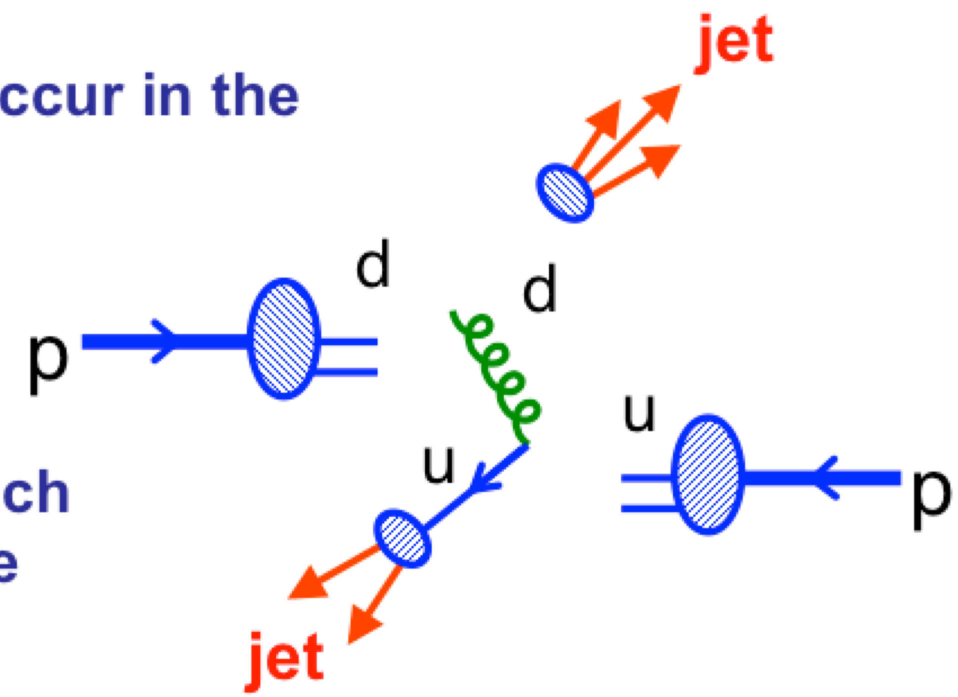
- Consider the process $u+d \to u+d$ which can occur in the high energy proton-proton scattering
- There are nine possible colour configurations of the colliding quarks which are all equally likely.
- Need to determine the average matrix element which is the sum over all possible colours divided by the number of possible initial colour states



$$\langle |M_{fi}|^2 \rangle = \frac{1}{3} \cdot \frac{1}{3} \sum_{i,j,k,l=1}^{3} |M_{fi}(ij \to kl)|^2$$

- The colour average matrix element contains the average colour factor

$$\langle |C|^2 \rangle = \frac{1}{9} \sum_{i,j,k,l=1}^{3} |C(ij \to kl)|^2$$

- For $qq \to qq$

$rr \to rr$,.. $rb \to rb$,.. $rb \to br$,..

$$\langle |C|^2 \rangle = \frac{1}{9}\left[3 \times \left(\frac{1}{3}\right)^2 + 6 \times \left(-\frac{1}{6}\right)^2 + 6 \times \left(\frac{1}{2}\right)^2\right] = \frac{2}{9}$$

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Quark-antiquark scattering

- Previously derived the Lorentz Invariant cross section for $e^-\mu^- \to e^-\mu^-$ elastic scattering in the ultra-relativistic limit

QED

$$\frac{d\sigma}{dq^2} = \frac{2\pi\alpha^2}{q^4}\left[1+\left(1+\frac{q^2}{s}\right)^2\right]$$

- For $ud \to ud$ in QCD replace $\alpha \to \alpha_s$ and multiply by $\langle|C|^2\rangle$

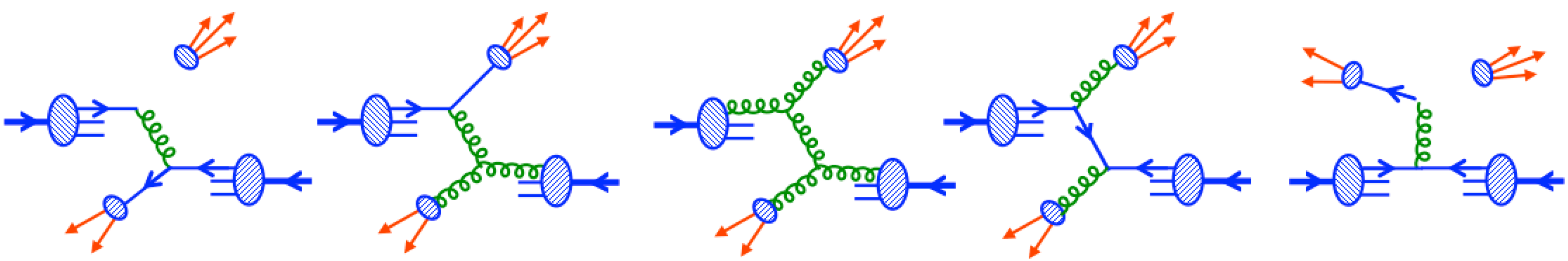
QCD

$$\frac{d\sigma}{dq^2} = \frac{2}{9}\frac{2\pi\alpha_S^2}{q^4}\left[1+\left(1+\frac{q^2}{\hat{s}}\right)^2\right]$$

Never see colour, but enters through colour factors. Can tell QCD is SU(3)

- Here $\hat{s}$ is the centre-of-mass energy of the quark-quark collision
- The calculation of hadron-hadron scattering is very involved, need to include parton structure functions and include all possible interactions

e.g. two jet production in proton-antiproton collisions

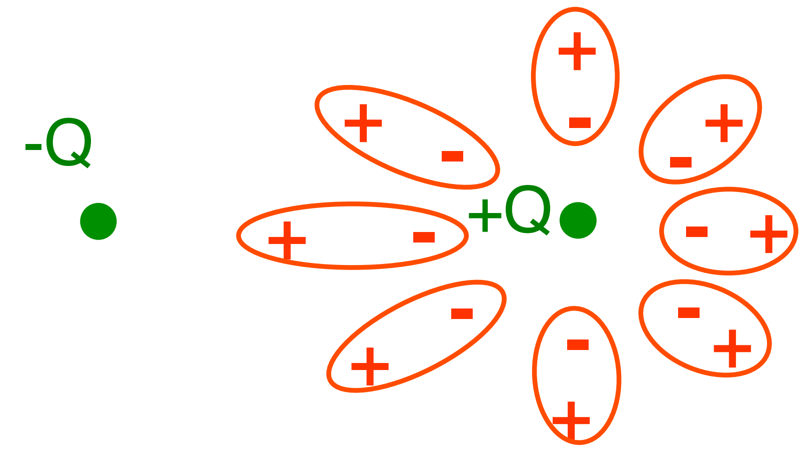


$q\bar{q} \to q\bar{q}$ $\quad qg \to qg$ $\quad gg \to gg$ $\quad q\bar{q} \to gg$ $\quad q\bar{q} \to q\bar{q}$

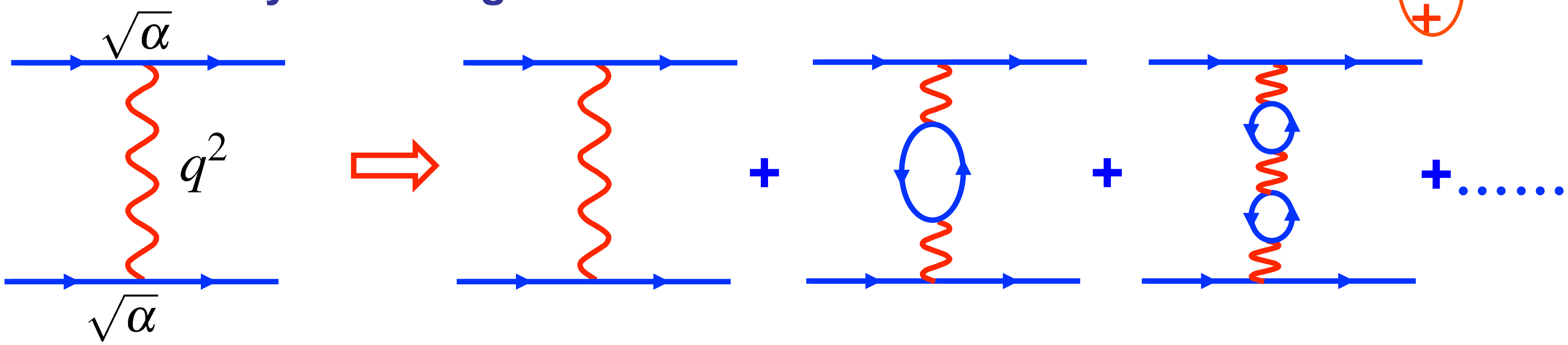
Running Coupling Constant

QED

- "bare" charge of electron screened by virtual e^+e^- pairs
- behaves like a polarizable dielectric

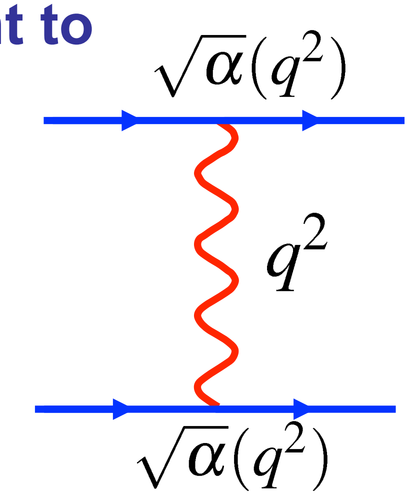


★ In terms of Feynman diagrams:



★ Same final state so add matrix element **amplitudes:** $M = M_1 + M_2 + M_3 + \ldots$

★ Giving an infinite series which can be summed and is equivalent to a single diagram with "running" coupling constant



$$\alpha(Q^2) = \alpha(Q_0^2) \Big/ \left[1 - \frac{\alpha(Q_0^2)}{3\pi} \ln\left(\frac{Q^2}{Q_0^2}\right)\right]$$

$$Q^2 \gg Q_0^2$$

Note sign

Running Coupling Constant

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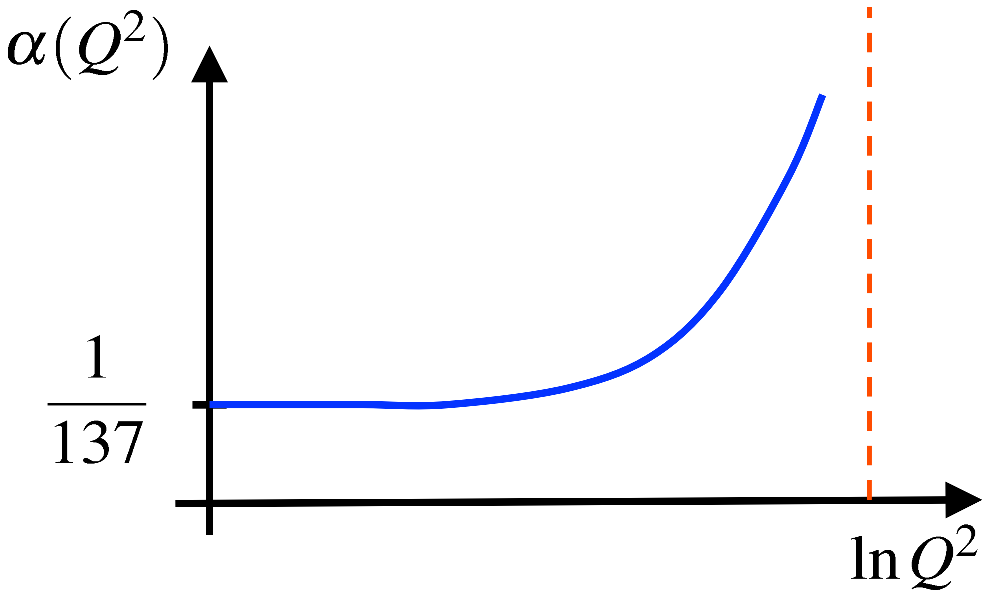
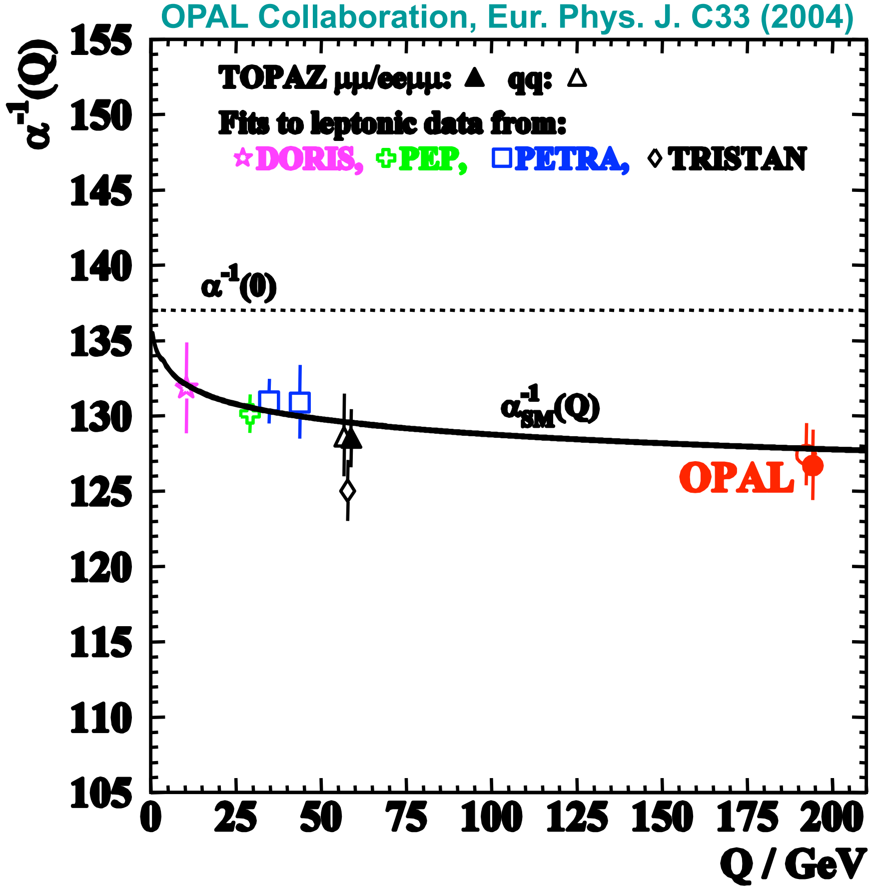


OPAL Collaboration, Eur. Phys. J. C33 (2004)



★ **Might worry that coupling becomes infinite at**

$$\ln\left(\frac{Q^2}{Q_0^2}\right) = \frac{3\pi}{1/137}$$

i.e. at $Q \sim 10^{26}\,\mathrm{GeV}$

- **But quantum gravity effects would come in way below this energy and it is highly unlikely that QED "as is" would be valid in this regime**

★ **In QED, running coupling increases very slowly**

- **Atomic physics:** $Q^2 \sim 0$

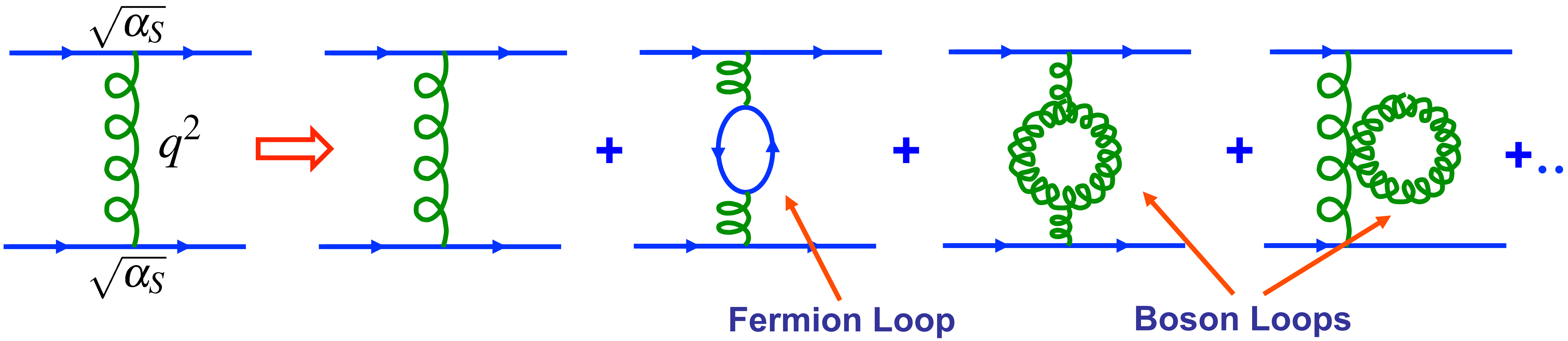
$$1/\alpha = 137.03599976(50)$$

- **High energy physics:**

$$1/\alpha(193\,\mathrm{GeV}) = 127.4 \pm 2.1$$

Running Coupling Constant

QCD Similar to QED but also have gluon loops



- ★ Remembering **adding amplitudes**, so can get negative interference and the sum can be smaller than the original diagram alone
- ★ Bosonic loops "interfere negatively"

$$\alpha_S(Q^2) = \alpha_S(Q_0^2) \Big/ \left[1 + B\alpha_S(Q_0^2)\ln\left(\frac{Q^2}{Q_0^2}\right)\right]$$

with $$B = \frac{11N_c - 2N_f}{12\pi}$$

N_c = no. of colours
N_f = no. of quark flavours

$N_c = 3;\ N_f = 6$ ➡ $B > 0$

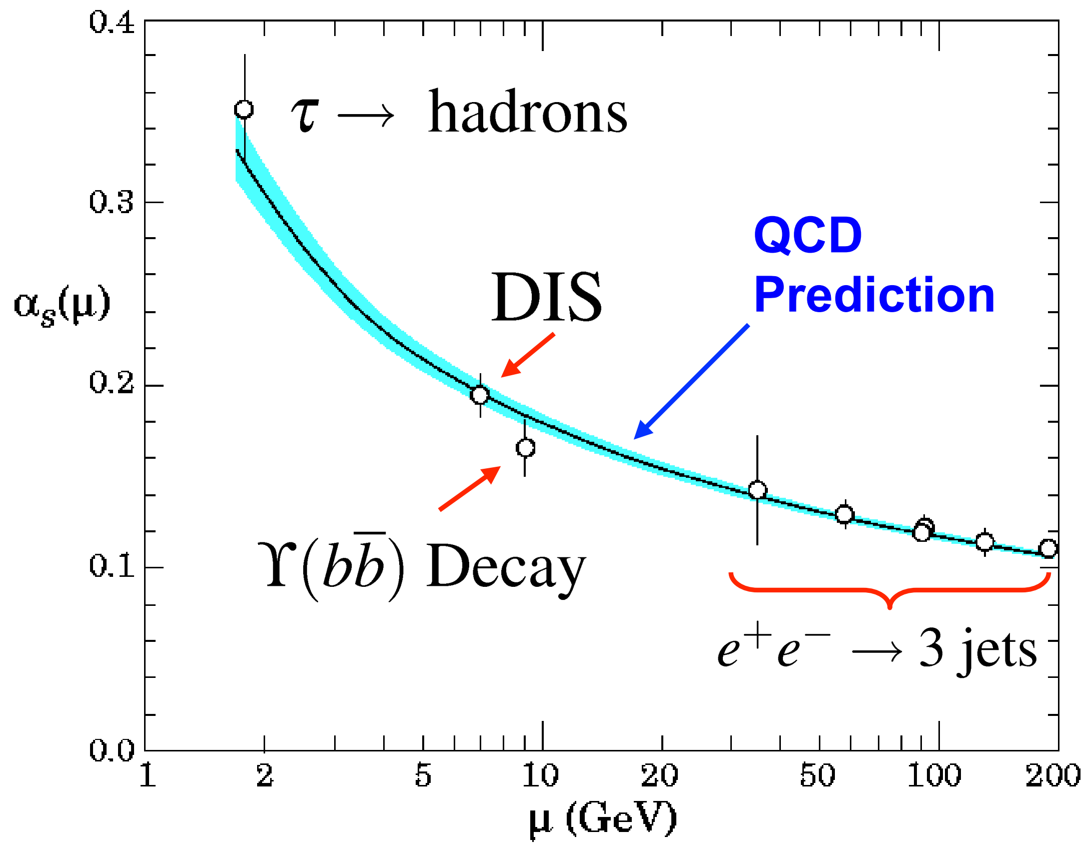
➡ **α_S decreases with Q^2**

Nobel Prize for Physics, 2004
(Gross, Politzer, Wilczek)

Running Coupling Constant

★ Measure α_s in many ways:
- jet rates
- DIS
- tau decays
- bottomonium decays
- +…

✶ As predicted by QCD, α_s decreases with Q^2



$\mu \approx Q$

★ At low Q^2: α_s is large, e.g. at $Q^2 = 1\,\mathrm{GeV}^2$ find α_s ~ 1

- Can't use perturbation theory ! This is the reason why QCD calculations at low energies are so difficult, e.g. properties hadrons, hadronisation of quarks to jets,…

★ At high Q^2 : α_s is rather small, e.g. at $Q^2 = M_Z^2$ find α_s ~ 0.12

➡ **Asymptotic Freedom**

- Can use perturbation theory and this is the reason that in DIS at high Q^2 quarks behave as if they are quasi-free (i.e. only weakly bound within hadrons)

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Running Coupling Constant

- ★ Superficially QCD very similar to QED
- ★ But gluon self-interactions are believed to result in colour confinement
- ★ All hadrons are colour singlets which explains why only observe

Mesons **Baryons**

- ★ A low energies $\alpha_S \sim 1$

➡ Can't use perturbation theory !

Non-Perturbative regime

- ★ Coupling constant runs, smaller coupling at higher energy scales

$$\alpha_S(100\,\mathrm{GeV}) \sim 0.1$$

➡ Can use perturbation theory

Asymptotic Freedom

- ★ **Where calculations can be performed, QCD provides a good description of relevant experimental data**