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General Relativity
Sheet 7

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Prof. P. Jetzer

M. Haney, S. Tiwari, A. Boitier, M. Ebersold, A. Dei, L. Fritz
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Exercise 1 [Commutator of vector fields]

(i) Verify that the commutator

$$[v,w](f) = v[w(f)] - w[v(f)] \tag{1}$$

where v and w are smooth vector fields, and f is a function, satisfies the linearity and Leibnitz properties, and is hence a vector field.

(ii) Let X, Y, and Z be vector fields on a manifold. Verify that their commutator satisfies the *Jacobi* identity, namely, that

$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0. \tag{2}$$

(iii) Show that in any coordinate basis, the components of the commutator of two vector fields v and w are given by

$$[v,w]^\mu = v^\nu \partial_\nu w^\mu - w^\nu \partial_\nu v^\mu. \tag{3}$$

Exercise 2 [Parallel transport]

Consider on $\mathbb{R}^2$ the connection defined by

$$\Gamma^1{}_{11} = x^1(p), \quad \Gamma^1{}_{12} = 1, \quad \Gamma^2{}_{22} = 2x^2(p),$$

where $p \in \mathbb{R}^2$ and the other Cristoffel symbols vanish.

Let γ be the straight curve in $\mathbb{R}^2$ defined by $\gamma(\lambda) = (\lambda, 0)$, $\lambda \in [0,1]$.

a) Compute the vector $v(\lambda)$, where $v(0) = \partial_2$, parallel transported along γ.

b) Write the geodesic equation (autoparallel transport) of this connection for a general curve γ.

Exercise 3 [Tensor Derivations]

A *tensor derivation* on a differentiable manifold M is a set $\{\mathscr{D}^r_s\}$ of $\mathbb{R}$-linear maps of $T^r_s(M)$ (the set of tensor fields of type (r,s) on M) into itself, such that (dropping the indices r,s):

(i) $\mathscr{D}\left(S_1 \otimes S_2\right) = (\mathscr{D}S_1) \otimes S_2 + S_1 \otimes \mathscr{D}S_2$

(ii) $\mathscr{D}$ commutes with contractions.

(iii) $\mathscr{D}f = X(f)$, with a well-defined vector field X associated to $\mathscr{D}$.

a) Prove that $\mathscr{D}$ can be restricted to any open set $U \subset M$.

   Hint: What are the conditions for $\mathscr{D}$ to be local? Show that $\mathscr{D}$ at a point p only depends on the (infinitesimal) neighborhood of p.

b) Derive the following formula:

$$\mathscr{D}S\left(X^1,\ldots,X^s,\omega_1,\ldots,\omega_r\right) = \mathscr{D}\left[S\left(X^1,\ldots,\omega_r\right)\right] - S\left(\mathscr{D}X^1,X^2,\ldots,\omega_r\right) - \ldots - S\left(X^1,\ldots,\mathscr{D}\omega_r\right),$$

   where X^i are vector fields and ω_j are covector fields.

c) Show that $\mathscr{D}$ is uniquely determined if its action on functions and vector fields is known.

The Lie derivative and the covariant derivative are both examples of tensor derivations. We have that:

- $L_X f = \nabla_X f = Xf$
- $L_X Y = [X, Y]$
- $\nabla_X \frac{\partial}{\partial x^j} = X^i \, \Gamma^k_{ij} \, \frac{\partial}{\partial x^k}$ (in a local coordinate system)

Use these relations and the results from b) and c) to calculate in local coordinates:

d) $(L_X Y)^i$ and $(L_X\omega)_i$

e) $(\nabla_X Y)^i$ and $(\nabla_X\omega)_i$

f) $(\nabla_X T)^i{}_j$ where $T = T^i{}_j \, \frac{\partial}{\partial x^i} \otimes dx^j$ is a $(1,1)$ tensor.