Elementary Particle Theory – PHY452

Fall Semester 2020

Exercise Sheet 7

Exercise 1: Soft-collinear radiation

Let us consider the process $e^+e^- \to \gamma^* \to q\bar{q}$ and derive the soft radiative corrections to this process, as illustrated in Fig. 1.

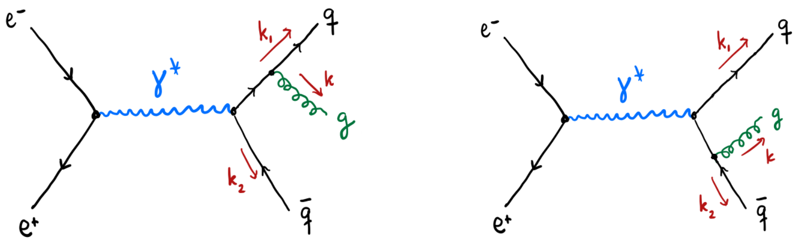

Figure 1: *Soft gluon correction to $e^+e^- \to q\bar{q}$ scattering.*

i) We denote the matrix element for the leading process by $\mathcal{M}_0$. Show that the the matrix element $\mathcal{M} \equiv \mathcal{M}(e^+e^- \to \gamma^* \to gq\bar{q})$ becomes, in the soft limit,

$$\mathcal{M} = \mathcal{M}_0 \, \varepsilon_\alpha(k,\lambda) \, J^{a\,\alpha}_{ij} \,, \tag{1}$$

where ε denotes the gluon polarization, i, j, a are color indices, and J is defined by

$$J^{a\,\alpha}_{ij} = g_S \, (T^a)_{ij} \left[\frac{k_2^\alpha}{k_2 \cdot k} - \frac{k_1^\alpha}{k_1 \cdot k} \right] . \tag{2}$$

Show that $J^{a\,\alpha}_{ij} \cdot k_\alpha = 0$ (i.e. the current J is conserved).

ii) Show that the averaged squared matrix element $|\mathcal{M}|^2$ factorizes into a soft term that multiplies the squared matrix element of the leading process $e^+e^- \to q\bar{q}$ as follows,

$$\overline{\sum} |\mathcal{M}|^2 = -g_S^2 C_F \left[\frac{k_1^2}{(k_1 \cdot k)^2} - \frac{2\,k_1 \cdot k_2}{(k_1 \cdot k)(k_2 \cdot k)} + \frac{k_2^2}{(k_2 \cdot k)^2} \right] \overline{\sum} |\mathcal{M}_0|^2 \,. \tag{3}$$

Discuss the behaviour for the different terms in Eq. (3) for soft/collinear gluons.

Exercise 2: IR safety

In QFT we encounter infrared divergences as a result of soft or collinear emissions. Quantities that qualify as physical observables, i.e. which could be potentially measured in experiments, must be defined in a so-called infrared-safe way. In accordance with the Kinoshita-Lee-Nauenberg-Theorem, an IR safe observable $\mathcal{O}$ satisfies the following conditions

$$\textit{soft emission}: \quad \mathcal{O}_{n+1}(p_1, \ldots, p_n, 0) \to \mathcal{O}_n(p_1, \ldots, p_n) \,, \tag{4}$$

$$\textit{collinear splitting}: \quad \mathcal{O}_{n+1}(p_1, \ldots, \lambda p_n, (1-\lambda) p_n) \to \mathcal{O}_n(p_1, \ldots, p_n), \quad \lambda \in [0,1] \,, \tag{5}$$

where n denotes the number of partons. These formulas express the stability of an observable in the limit of soft or collinear emission. Consider the following observables and discuss their IR safety:

1. Total sum of energy of quarks and anti-quarks.
2. Total number of QCD partons.
3. Total sum of energy of all QCD partons.
4. Direction of the most energetic QCD parton.
5. Energy flow into a cone defined as all final-state particles with momenta $\vec{p}$ that are separated by an angle $\theta < \theta_{\text{cone}}$ with respect to the cone axis $\vec{n}$.
6. Thrust $\sim \max_{\vec{n}} \frac{\sum_i |\vec{p}_i \cdot \vec{n}|}{\sum_i |\vec{p}_i|}$, $|\vec{n}| = 1$
7. Sphericity $\sim \min_{\vec{n}} \frac{\sum_i (\vec{p}_i \times \vec{n})^2}{\sum_i \vec{p}_i^2}$, $|\vec{n}| = 1$

Exercise 3: Running coupling

The goal of this exercise is to understand qualitatively the interplay of renormalization and scale dependence in QFT. To this end, consider the production of charged fermions through an s-channel exchange of a photon,

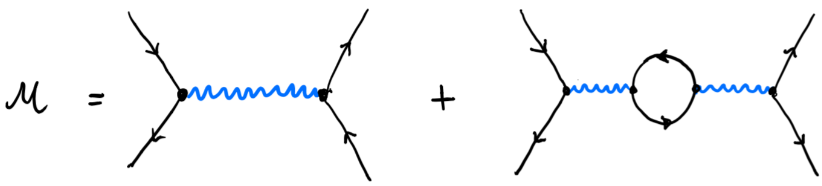

Figure 2: *Loop correction to quark-pair production.*

where we restrict ourselves to fermionic loops, i.e. not including any vertex corrections. When renormalizing a theory, we typically start from a Lagrangian expressed in terms of bare couplings as follows [1]

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} \left(i\gamma^\mu D_\mu - m \right) \psi \,, \tag{6}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,, \qquad D_\mu = \partial_\mu - i g_{\text{B}} A_\mu \,, \tag{7}$$

Let us assume that the bare amplitude in Fig. 2 can be written as

$$\mathcal{M} = A \frac{g_{\text{B}}^2}{q^2} \left[1 + \Pi(q^2) \right] , \qquad \text{with} \qquad \Pi(q^2) = \frac{g_{\text{B}}^2}{12\pi^2} \log \frac{q^2}{\Lambda_{\text{UV}}^2} \,, \tag{8}$$

where q is the exchanged momentum, A is some prefactor (not relevant) and Λ_{UV} takes the role of a UV cutoff that needs to be introduced in order to regularize the loop integral.

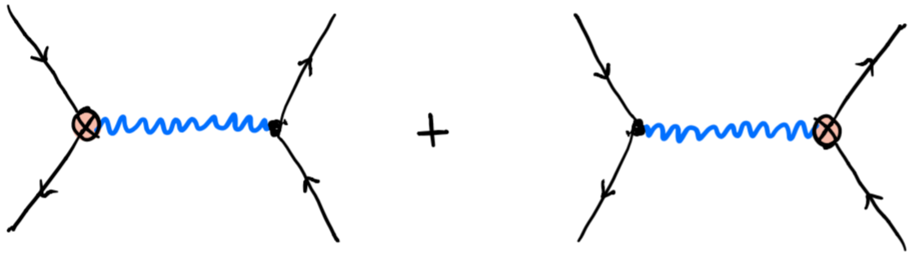

Figure 3: *Counter-terms for the process in Fig. 2. The crosses represent the counterterm Feynman rule (of order $\mathcal{O}(\delta g)$) that emerges as a result of (9) in the covariant derivative (7).*

i) The bare interactions in Eq. (7) can be split into two vertices that are proportional to g and $g\,\delta Z_g$,

$$g_{\rm B} = g\,(1 + \delta Z_g)\,, \tag{9}$$

where g and δZ_g denote the renormalized coupling and counterterm, respectively. Compute the amplitude for the counterterms in Fig. 3 in terms of g and δZ_g by keeping $\mathcal{O}(g^4)$ terms (formally $\delta Z_g = \mathcal{O}(g^2)$).

ii) Impose the renormalization condition

$$\mathcal{M}\big|_{q^2=q_0^2} \overset{!}{=} A\,\frac{g^2}{q_0^2}, \tag{10}$$

which defines the physical coupling $\alpha(q_0^2) = g^2/4\pi$ and solve it for δZ_g. The renormalization condition (10) relates the renormalised parameter g to the effective strength of the interaction in an experimental.

iii) Derive the running of α by relating the two values $\alpha(q^2)$ and $\alpha(q_0^2)$ at different energies q and q_0, respectively.

[1]Normally, also fields are subject to renormalization, but for this exercise we will not be renormalizing them.