MMP 2
Solutions 4

FS 19
Prof. Ph. Jetzer

Exercise 1

$$
\begin{aligned}
f(z) &= \frac{iz^2+1}{1-iz^2} \\
z^2 &= (x+iy)^2 = x^2-y^2+i2xy = u+iv \\
u &= x^2-y^2 \\
v &= 2xy
\end{aligned}
$$

a) $z \in U_< \Leftrightarrow |f(z)| < 1$

$$
\begin{aligned}
1 &> |f(z)| \\
1 &> |f(z)|^2 \\
1 &> \underbrace{\frac{(iz^2+1)(-i\overline{z}^2+1)}{(1-iz^2)(1+i\overline{z}^2)}}_{(1-iz^2)(1+i\overline{z}^2)=|1-iz^2|^2>0}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow (1-iz^2)(1+i\overline{z}^2) &> (iz^2+1)(-i\overline{z}^2+1) \\
1-iz^2+i\overline{z}^2+z^2\overline{z}^2 &> 1+iz^2-i\overline{z}^2+z^2\overline{z}^2 \\
i(-u-iv+u-iv) &> i(u+iv-u+iv) \\
2v &> -2v \\
4v &> 0 \\
2xy &> 0
\end{aligned}
$$

$z \in U_< \Leftrightarrow Im(z^2) > 0 \Leftrightarrow |f(z)| < 1$

b) Analoguely to 1a):

$2xy = 0$

$z \in U_= \Leftrightarrow Im(z^2) = 0 \Leftrightarrow |f(z)| = 0$

c) Analoguely to 1a):

$2xy > 0$

$z \in U_> \Leftrightarrow Im(z^2) < 0 \Leftrightarrow |f(z)| > 1$

And: $z \neq \pm\frac{1}{\sqrt{2}}(1-i)$

(Singularity)

Exercise 2

a) circle C , centred at $z = 0$

$$\begin{aligned} z &= re^{i\phi} \\ dz &= ire^{i\phi}d\phi \\ \frac{1}{2\pi i}\oint_C \frac{dz}{z} &= \frac{1}{2\pi i}\int_{-\pi}^{\pi} \frac{ire^{i\phi}d\phi}{re^{i\phi}} \\ &= \frac{1}{2\pi i}\int_{-\pi}^{\pi} id\phi \\ &= 1 \end{aligned}$$

b) C circle of radius 2, centred at $z = 0$

Cauchy: $\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}dz = f(z_0)$, z_0 within C

$$\begin{aligned} \oint_C \frac{\sin z}{2z-\pi}dz &= \oint_C \frac{1}{2}\frac{\sin z}{z-\frac{\pi}{2}} \\ &= \frac{2\pi i}{2}\sin\frac{\pi}{2} \\ &= \pi i \end{aligned}$$

c) square C , vertices ± 1 , $\pm i$

$$\oint_C \frac{e^{3z}}{(z-\ln 2)^4}dz$$

$$\begin{aligned} f^{(n)}(z_0) &= \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}} \\ f(z) &= e^{3z} \\ f^{(3)}(z) &= 27e^{3z} \\ \Rightarrow \oint_C \frac{e^{3z}}{(z-\ln 2)^4}dz &= \frac{2\pi i}{3!}\cdot 27e^{3\ln 2} \\ &= 72\pi i \end{aligned}$$

d) Circle C, radius 1, $a, b \in \mathbb{R}$, $0 < b < a$

$$
\begin{aligned}
z &= e^{i\theta} = \cos\theta + i\sin\theta \\
z^{-1} &= e^{-i\phi} = \cos\theta - i\sin\theta \\
\Rightarrow \cos\theta &= \frac{1}{2}(z + \frac{1}{z}) \\
d\theta &= -\frac{i}{z}dz \\
\oint_C \frac{d\theta}{a + b\cos\theta} &= \oint_C \frac{dz}{iz(a + \frac{b}{2}(z + \frac{1}{z}))} \\
&= \oint_C \frac{dz}{i(az + \frac{b}{2}z^2 + \frac{b}{2})} \\
&= \oint_C \frac{2dz}{ib(z^2 + \frac{2a}{b}z + 1)}
\end{aligned}
$$

Let $c_\pm = \frac{-\frac{2a}{b} \pm \sqrt{\frac{4a^2}{b^2} - 4}}{2} = \frac{1}{b}(-a \pm \sqrt{a^2 - b^2})$

c_+ is in the circle: $|c_+| = |\frac{1}{b}(-a + \sqrt{a^2 - b^2})| = |\frac{1}{b}(a - \sqrt{a^2 - b^2})|$

Since $b < a \Rightarrow 2b^2 < 2ab$

$\Rightarrow (a - b)^2 = a^2 - 2ab + b^2 < a^2 - 2b^2 + b^2 = a^2 - b^2$

$a - b < \sqrt{a^2 - b^2} \Rightarrow a - \sqrt{a^2 - b^2} < b$

$\Rightarrow |\frac{a - \sqrt{a^2 - b^2}}{b}| = |c_+| < 1$

$$
\begin{aligned}
\Rightarrow \oint_C \frac{d\theta}{a + b\cos\theta} &= \oint_C \frac{2dz}{ib(z - c_-)(z - c_+)} \\
&= 2\pi i \cdot \frac{2}{ib} \cdot \frac{1}{c_+ - c_-} \\
&= \frac{4\pi}{b}\left(\frac{b}{-a + \sqrt{a^2 - b^2} + a + \sqrt{a^2 - b^2}}\right) \\
&= \frac{2\pi}{\sqrt{a^2 - b^2}}
\end{aligned}
$$

Exercise 3

Geometric series:

$r \neq 1 : a\frac{1 - r^n}{1 - r} = \sum_{k=0}^{n-1} ar^k$

$r < 1 : \sum_{k=0}^{\infty} ar^k = \frac{a}{1 - r}$

a) $f(z) = \frac{1}{1+z^2}$ about $z = 0$

$$\begin{aligned} f(z) &= \frac{1}{(1+iz)(1-iz)} \\ &= \frac{\frac{1}{2}(1+iz+1-iz)}{(1+iz)(1-iz)} \\ &= \frac{1}{2}\left[\frac{1}{(1+iz)} + \frac{1}{(1-iz)}\right] \end{aligned}$$

Case $|z| < 1$:

$$\begin{aligned} f(z) &= \frac{1}{2}\left[\sum_{n=0}^{\infty}(iz)^n + \sum_{n=0}^{\infty}(-iz)^n\right] \\ &= \frac{1}{2}\left[\sum_{n=0}^{\infty} i^n(z^n + (-z)^n)\right] \\ &= \frac{1}{2}\sum_{n=0}^{\infty} 2i^{2n}z^{2n} \\ &= \sum_{n=0}^{\infty}(-1)^n z^{2n} \end{aligned}$$

Case $|z| > 1$:

$$\begin{aligned} f(z) &= \frac{1}{2iz}\left[\frac{-1}{1-\frac{1}{iz}} + \frac{1}{1+\frac{1}{iz}}\right] \\ &= \frac{1}{2iz}\left[-\sum_{n=0}^{\infty}\left(\frac{-1}{iz}\right)^n + \sum_{n=0}^{\infty}\left(\frac{1}{iz}\right)^n\right] \\ &= \frac{1}{2iz}\left[\sum_{n=0}^{-\infty} i^n(z^n - (-z)^n)\right] \\ &= \frac{1}{2iz}\sum_{n=0}^{-\infty} i^{2n-1}\cdot 2z^{2n-1} \\ &= \sum_{n=0}^{-\infty} i^{2n-2}z^{2n-2} \\ &= \sum_{n=0}^{-\infty}(-1)^n z^{2n-2} \end{aligned}$$

b) $f(z) = \frac{z+2}{z+3} = \frac{z+3-1}{z+3} = 1 - \frac{1}{z+3}$ about $z = -1$

Case $|z+1| < 2$:

$$
\begin{aligned}
f(z) &= 1 - \frac{1}{z+3} \\
&= 1 - \frac{1}{2}\frac{1}{\frac{z+1}{2}+1} \\
&= 1 - \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{z+1}{2}\right)^n \\
&= \frac{1}{2} + \sum_{n=1}^{\infty}(-1)^{n+1}2^{-n-1}(z+1)^n
\end{aligned}
$$

Case $|z+1| > 2$

$$
\begin{aligned}
f(z) &= 1 - \frac{1}{z+1}\frac{1}{1+\frac{2}{z+1}} \\
&= 1 - \frac{1}{z+1}\sum_{n=0}^{\infty}\left(\frac{-2}{z+1}\right)^n \\
&= 1 - \sum_{n=0}^{\infty}2^n(z+1)^{-n-1}(-1)^n \\
&= 1 + \sum_{n=0}^{-\infty}2^{-n}(z+1)^{n-1}(-1)^{n+1} \\
&= 1 + \sum_{n=-1}^{-\infty}2^{-n-1}(z+1)^n(-1)^n
\end{aligned}
$$

c) Binomial Theorem:

$$
\begin{aligned}
(1+z)^\alpha &= \sum_{n=0}^{\infty}\binom{a}{n}z^n \\
\binom{a}{n} &= \frac{a!}{(a-n)!n!} \\
\binom{b}{c} &= (-1)^c\binom{c-b-1}{c}
\end{aligned}
$$

$f(z) = \frac{1}{z(z-1)(z-2)^2}$ about $z = 0$

$$
\begin{aligned}
f(z) &= \frac{1}{z(z-1)(z-2)^2} \\
&= \frac{1}{z}\left(\frac{1}{z-1} - \frac{1}{z-2} + \frac{1}{(z-2)^2}\right)
\end{aligned}
$$

Case $|z| < 1$:

$$
\begin{aligned}
f(z) &= \frac{1}{z}\left(-\frac{1}{1-z} + \frac{1}{2}\frac{1}{1-\frac{z}{2}} + \frac{1}{4}\frac{1}{(1-\frac{z}{2})^2}\right) \\
&= \frac{1}{z}\left[-\sum_{n=0}^{\infty} z^n + \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n + \frac{1}{4}\sum_{n=0}^{\infty}\binom{-2}{n}\left(-\frac{z}{2}\right)^n\right] \\
&= \sum_{n=0}^{\infty} z^{n-1}\left(-1 + 2^{-n-1} + 2^{-n-2}\binom{2}{n}(-1)^n\right) \\
\binom{-2}{n} &= (-1)^n\binom{n+2-1}{n} = (-1)^n\binom{n+1}{n} = (-1)^n\frac{(n+1)!}{1!n!} = (-1)^n(n+1) \\
\Rightarrow f(z) &= \sum_{n=0}^{\infty} z^{n-1}\left(-1 + 2^{-n-1} + 2^{-n-2}(n+1)\right) \\
&= \sum_{n=-1}^{\infty} z^n(-1 + 2^{-n-2} + 2^{-n-3}(n+2))
\end{aligned}
$$

Case $1 < |z| < 2$:

$$
\begin{aligned}
f(z) &= \frac{1}{z}\left(\frac{1}{z}\frac{1}{(1-\frac{1}{z})} + \frac{1}{2}\frac{1}{1-\frac{z}{2}} + \frac{1}{4}\frac{1}{(1-\frac{z}{2})^2}\right) \\
&= \frac{1}{z}\left[\frac{1}{z}\sum_{n=0}^{\infty} z^{-n} + \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n + \frac{1}{4}\sum_{n=0}^{\infty}(n+1)(-1)^n\left(-\frac{z}{2}\right)^n\right] \\
&= \sum_{n=0}^{\infty} z^{-n-2} + \sum_{n=0}^{\infty} z^{n-1}\left(2^{-n-1} + 2^{-n-2}(n+1)\right) \\
&= \sum_{n=-2}^{-\infty} z^n + \sum_{n=-1}^{\infty} z^n\left(2^{-n-2} + (n+2)2^{-n-3}\right)
\end{aligned}
$$

Case $|z| > 2$:

$$
\begin{aligned}
f(z) &= \frac{1}{z}\left(\frac{1}{z}\frac{1}{(1-\frac{1}{z})} - \frac{1}{z}\frac{1}{(1-\frac{2}{z})} + \frac{1}{z^2}\frac{1}{(1-\frac{2}{z})^2}\right) \\
&= \frac{1}{z}\left(\frac{1}{z}\sum_{n=0}^{\infty} z^{-n} - \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n + \frac{1}{z^2}\sum_{n=0}^{\infty}(-1)^n(n+1)\left(-\frac{2}{z}\right)^n\right) \\
&= \sum_{n=0}^{\infty} z^{-n-2} - \sum_{n=0}^{\infty} 2^n z^{-n-2} + \sum_{n=0}^{\infty} 2^n(n+1)z^{-n-3} \\
&= \sum_{n=2}^{\infty}(1-2^{n-2})z^{-n} + \sum_{n=3}^{\infty} 2^{n-3}(n-2)z^{-n} \\
&= \sum_{n=3}^{\infty}\left(1 - 2^{n-2} + 2^{n-3}(n-2)\right)z^{-n} \\
&= \sum_{n=-3}^{-\infty}\left(1 - 2^{-n-2} - 2^{n-3}(-n+2)\right)z^n
\end{aligned}
$$

d) $\frac{\sin \pi z}{4z^2-1}$ about $z = \frac{1}{2}$

$$\begin{aligned} f(z) &= \frac{\sin \pi z}{4z^2-1} \\ &= \frac{\sin \pi z}{(2z+1)(2z-1)} \\ &= \frac{\sin \pi z}{2}\left[\frac{-1}{2z+1} + \frac{1}{2z-1}\right] \end{aligned}$$

Case $|z - \frac{1}{2}| < 1$:

$$\begin{aligned} f(z) &= \frac{\sin\left(\left(\pi(z-\frac{1}{2}) + \frac{\pi}{2}\right)\right.}{4}\left[\frac{-1}{(z-\frac{1}{2})+1} + \frac{1}{z-\frac{1}{2}}\right] \\ &= \frac{\cos\left(\pi(z-\frac{1}{2})\right)}{4}\left[-\sum_{n=0}^{\infty}\left(z-\frac{1}{2}\right)^n(-1)^n + \left(z-\frac{1}{2}\right)^{-1}\right] \\ &= \frac{1}{4}\left[\sum_{n=0}^{\infty}\frac{(-1)^n\left(\pi(z-\frac{1}{2})\right)^{2n}}{(2n)!}\right]\left[\sum_{n=-1}^{\infty}\left(z-\frac{1}{2}\right)^n(-1)^{n+1}\right] \\ &= \frac{1}{4}\left[\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n}\left(z-\frac{1}{2}\right)^{2n}}{(2n)!}\right]\left[\sum_{n=-1}^{\infty}\left(z-\frac{1}{2}\right)^n(-1)^{n+1}\right] \\ &= \frac{1}{4}\left[\frac{1}{z-\frac{1}{2}} - 1 + \left(z-\frac{1}{2}\right)\left(1-\frac{\pi^2}{2}\right) + ...\right] \end{aligned}$$

Case $|z - \frac{1}{2}| > 1$:

$$\begin{aligned} f(z) &= \frac{\sin \pi z}{2}\left[\frac{-1}{1+\frac{1}{z-\frac{1}{2}}}(z-\frac{1}{2})^{-1} + \frac{1}{z-\frac{1}{2}}\right] \\ &= \frac{\cos\left(\pi(z-\frac{1}{2})\right)}{4}\left[-\sum_{n=0}^{\infty}\left(z-\frac{1}{2}\right)^{-n}(-1)^n + \left(z-\frac{1}{2}\right)^{-1}\right] \\ &= \frac{1}{4}\left[\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n}\left(z-\frac{1}{2}\right)^{2n}}{(2n)!}\right]\left[\sum_{n=0}^{-\infty}\left(z-\frac{1}{2}\right)^n(-1)^{n+1} + \left(z-\frac{1}{2}\right)^{-1}\right] \end{aligned}$$