MMP II
Exam Solutions

FS 19
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Exercise 1

a)

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz}) \overset{!}{=} 2 \quad \text{substitute: } t = e^{iz}$$
$$t - \frac{1}{t} = 4i \quad \Leftrightarrow \quad t^2 - 4it - 1 = (t-2i)^2 + 3 = 0$$
$$\Rightarrow \quad t = \pm\sqrt{-3} + 2i = i(2 \pm \sqrt{3}) = e^{iz}$$
$$\Rightarrow \quad z = \frac{1}{i}\ln(i(2\pm\sqrt{3})) = \underline{\underline{\frac{\pi}{2} - i\ln(2\pm\sqrt{3})}} \qquad \text{1 pt}$$

check:

$$\sin\left(\frac{\pi}{2} - i\ln(2\pm\sqrt{3})\right) = \frac{1}{2i}(e^{i\frac{\pi}{2}+\ln(2\pm\sqrt{3})} - e^{-i\frac{\pi}{2}-\ln(2\pm\sqrt{3})}) = \frac{1}{2i}\underbrace{\left(i(2\pm\sqrt{3}) + \frac{i}{(2\pm\sqrt{3})}\right)}_{t-\frac{1}{t}=4i} = 2 \quad \checkmark$$

b) The property 3) holds in the whole complex plane. Therefore, if we consider the region $\mathrm{Re}z < 1$, the property 1) ensure us that $f(1-z)$ is analytic, since $\mathrm{Re}(1-z) > 0$. This implies that all the singularities of $\sin^{-1}\pi z$ within this region are the singularities of $f(z)$. Thus, $f(z)$ has simple poles at

$$\sin \pi z_k = 0\,, \quad \text{with} \qquad \mathrm{Re}z < 1 \qquad \Rightarrow \qquad z_k = 0, -1, -2, \ldots$$

In order to compute the residue of $f(z)$ at these singular points, we observe that, for $z_k = -k$, $k \in \mathbb{N}$,

$$\begin{aligned}\mathrm{Res} f(z)f(1-z)\big|_{z_k} &= \lim_{z\to -k}(z+k)f(z)f(1-z)\\ &= f(1+k)\lim_{z\to -k}(z+k)f(z)\\ &= k!\mathrm{Res} f(z)\big|_{z_k}\,,\end{aligned}$$

where we have used the regularity of $f(1-z)$, as well as the property 2). Finally, thanks to 3), we have

$$\begin{aligned}
\mathrm{Res} f(z)\big|_{z_k} =& \frac{1}{k!}\mathrm{Res}\frac{\pi}{\sin \pi z}\Big|_{z_k} \\
=& \frac{\pi}{k!}\lim_{z\to -k}\frac{(z+k)}{\sin \pi z} \\
=& \frac{1}{k!}\frac{1}{\cos \pi z} \\
=& \frac{(-1)^k}{k!}\,.
\end{aligned}$$

c) In order to derive the Laurent series of $f(z)$ we begin by partial fraction the functions as

$$f(z) = \frac{A}{z^2} + \frac{B}{z} + \frac{C}{z+3}\,.$$

by demanding that

$$(C+B)z^2 + (A+3B)z + 3A = 4 + z\,,$$

we find $A = 4/3$, $B = -C = -1/9$. Therefore, we have

$$f(z) = \frac{4}{3}\frac{1}{z^2} - \frac{1}{9}\frac{1}{z} + \frac{1}{9(z+3)}\,.$$

The first two terms, which display the singular behaviour of $f(z)$ in z=0, are already part of the Laurent expansion of the function. Therefore, we can just focus on the last term, which is regular in $z = 0$. We observe that

$$\begin{aligned}
\frac{1}{9(z+3)} =& \frac{1}{27\left(1+\frac{z}{3}\right)} \\
\underset{|z|<3}{=}& \frac{1}{27}\sum_{k=0}^{\infty}\left(-\frac{z}{3}\right)^k \\
=& \frac{1}{27}\sum_{k=0}^{\infty}\frac{(-1)^k}{3^k}z^k\,.
\end{aligned}$$

Hence, we obtain

$$f(z) = \frac{4}{3}\frac{1}{z^2} - \frac{1}{9}\frac{1}{z} + \frac{1}{27}\sum_{k=0}^{\infty}\frac{(-1)^k}{3^k}z^k\,,$$

and, from the coefficient of z^{-1}, we can immediately read off $\mathrm{Res} f(z)|_{z=0} = -1/9$.

Exercise 2

a)
- The function has simple zeroes corresponding to the zeroes of $\sin \pi z$:

$$\sin \pi z_n = 0\,, \qquad z_{\pm n} = \pm n\,, \quad n \in \mathbb{N}\,.$$

- The function has simple poles corresponding to the zeroes of $\cos \pi z$:

$$\cos \pi z_k = 0\,, \qquad z_{\pm k} = \pm \frac{(2k+1)}{2}\,, \quad k \in \mathbb{N}\,.$$

  The residue at z_k is given by

$$\begin{aligned}
\mathrm{Res} f(z)\big|_{z=z_k} &= \lim_{z \to z_{\pm k}} f(z)(z - z_{\pm k}) = \\
&\frac{\sin \pi z_{\pm k}}{(z-a)^2} \lim_{z \to z_{\pm k}} \frac{(z - z_{\pm k})}{\cos \pi z} \\
&\overset{l'H\hat{o}pital}{=} \frac{\sin \pi z_{\pm k}}{(z-a)^2} \lim_{z \to z_{\pm k}} \frac{-1}{\pi \sin \pi z} \\
&= -\frac{-1}{\pi (z_{\pm k} - a)^2} = -\frac{1}{\pi (k + \frac{1}{2} \mp a)^2}\,.
\end{aligned}$$

- The function has an additional double pole at $z = a$ and its residue is

$$\mathrm{Res} f(z)\big|_{z=a} = \lim_{z \to a} \frac{d}{dz} f(z)(z-a)^2 = \lim_{z \to a} \frac{d}{dz} \tan \pi z = \frac{\pi}{\cos^2 \pi a}\,.$$

- The point at infinity is an essential singularity of $f(z)$, as $z = \infty$ is an accumulation point of zeroes.

b) In order to evaluate the integral

$$I = \int_{-\infty + i\epsilon}^{\infty + i\epsilon} f(z) dz\,, \qquad \epsilon \in \mathbb{R}^+$$

we observe that, since $\tan(\pi z)$ is bounded, $f(z) \underset{|z| \to \infty}{\sim} \frac{1}{|z|^2}$ along every direction in the complex plane. Therefore, we can close the integration contour in both half-planes. If we close the contour in the upper plane $\mathrm{Im} z > \epsilon$, we immediately see that

$$I = \oint_{\Gamma_1} dz f(z) = 0\,,$$

since, Γ_1 does not contain any of the poles of $f(z)$.
On the other had, if we close the contour in the lower plane $\mathrm{Im} z < \epsilon$, the integral is written as the sum of the residues at every pole of $f(z)$,

$$\begin{aligned}
I = \oint_{\Gamma_2} dz f(z) &= -2\pi i \left[\mathrm{Res} f(z)|_{z=a} + \sum_{k=0}^{\infty} \mathrm{Res} f(z)|_{z=z_{\pm k}} \right] \\
&= -2i \left[\frac{\pi}{\cos^2 \pi a} - 4 \sum_{k=0}^{\infty} \left(\frac{1}{(2k+1+2a)^2} + \frac{1}{(2k+1-2a)^2} \right) \right]\,.
\end{aligned}$$

c) Of course, the two expressions must be identical. Therefore, by imposing the vanishing of I, we obtain the sum

$$\sum_{k=0}^{\infty}\left(\frac{1}{(2k+1+2a)^2}+\frac{1}{(2k+1-2a)^2}\right)=\frac{\pi^2}{4\cos^2\pi a}\,.$$

Although the result was obtained under the assumption $\mathrm{Im}a<0$, we can assume it to be valid for all the complex values of a for which the r.h.s. is finite (or it has a divergency that compensates exactly the one of the l.h.s.). For instance, if we set $a=0$, we obtain

$$\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}=\frac{\pi^2}{8}\,.$$

The second result that we are asked to prove can be obtained for $a=1/2$. However, due to apparent divergence of our expression, the limit $a\to 1/2$ must be taken with care on both sides. If we separate the $k=0$ term of the second sum, we have

$$\sum_{k=0}^{\infty}\frac{1}{(2k+2)^2}+\sum_{k=1}^{\infty}\frac{1}{(2k)^2}+\lim_{a\to 1/2}\frac{1}{(1-2a)^2}=\lim_{a\to 1/2}\frac{\pi^2}{4\cos^2\pi a}\,,$$

which, by shifting $k\to k-1$ in the first sum, can be rewritten as

$$\sum_{k=1}^{\infty}\frac{1}{k^2}=2\lim_{a\to 1/2}\left(\frac{\pi^2}{4\cos^2\pi a}-\frac{1}{(1-2a)^2}\right)\,,$$

If we now use

$$\cos\pi a=\cos\left((a-1/2)\,\pi+\pi/2\right)=-\sin\left((a-1/2)\,\pi\right)\,,$$

we can expand around $a=1/2$,

$$\cos\pi a=-\pi\left(a-\frac{1}{2}\right)+\frac{\pi^3}{6}\left(a-\frac{1}{2}\right)^3+\mathcal{O}\left((a-1/2)^5\right)\,.$$

and get the Laurent series

$$\begin{aligned}\frac{\pi^2}{4\cos^2\pi a}=&\frac{1}{4(a-1/2)^2}\left(1-\frac{\pi^2}{6}\left(a-\frac{1}{2}\right)^3+\mathcal{O}\left((a-1/2)^5\right)\right)^{-2}\\ =&\frac{1}{4(a-1/2)^2}\left(1+\frac{\pi^2}{3}\left(a-\frac{1}{2}\right)^3+\mathcal{O}\left((a-1/2)^5\right)\right)\\ =&\frac{1}{(1-2a)^2}+\frac{\pi^2}{12}+\mathcal{O}\left((a-1/2)^2\right)\,.\end{aligned}$$

The leading pole of such expansion cancels exactly the divergence at $k=0$ of the series, so that

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{1}{k^2}=&2\lim_{a\to 1/2}\left(\frac{\pi^2}{4\cos^2\pi a}-\frac{1}{(1-2a)^2}\right)\\ =&\lim_{a\to 1/2}\left(\frac{\pi^2}{6}+\mathcal{O}\left((a-1/2)^2\right)\right)\\ =&\frac{\pi^2}{6}\,.\end{aligned}$$

Exercise 3

The integral I_1 can be computed by means of the residue theorem through the change of variables

$$z = e^{2ix}\,, \qquad dx = -\frac{i}{2}\frac{dz}{z}\,, \qquad 1 + 4\sin^2 x = -\frac{z^2 - 3z + 1}{z}\,.$$

We observe that, if $x \in [0, 2\pi]$, the complex variable z spanns *twice* the unit circle centred in the origin C_1. Thus, we write

$$\begin{aligned} I_1 =& 2\oint_{C_1} \frac{dz}{z}\left(\frac{i}{2}\right)\frac{z}{z^2 - 3z + 1} \\ =& i\oint_{C_1} dz \frac{1}{(z - z_+)(z - z_-)}\,, \end{aligned}$$

with

$$z_\pm = \frac{3 \pm \sqrt{5}}{2}\,.$$

Only the simple pole z_- lies inside C_1, since $|z_+| > 1$. Hence, we have

$$\begin{aligned} I_1 =& 2\pi i \mathrm{Res} \frac{i}{(z - z_+)(z - z_-)}\bigg|_{z=z_-} \\ =& -2\pi \lim_{z\to z_-} \frac{1}{z - z_+} \\ =& \frac{2\pi}{\sqrt{5}}\,. \end{aligned}$$

After promoting $x \to z$, we observe that, due to the polydromy of $\sqrt[3]{z}$, the integrand of I_2 has a branch cut on the complex plane, with branching points at $z = 0$ and $z = \infty$. We can fix the Riemann sheet in such a way that the branch cut runs over the positiv real axis so that, once parametrized $z = \rho e^{i\theta}$, we have

$$\sqrt[3]{z} = \begin{cases} \sqrt[3]{x}, & \text{above the cut} \\ e^{i\frac{2}{3}\pi}\sqrt[3]{x}, & \text{below the cut} \end{cases}\,.$$

We can now consider the integral J along the closed contour Γ displayed in Figure **??** and decompose it as

$$J = \oint_\Gamma \frac{\sqrt[3]{z}}{1 + z^2} dz = \left(\int_{C_\epsilon} + \int_{C_R} + \int_{R_+} + \int_{R_-}\right)\frac{\sqrt[3]{z}}{1 + z^2} dz\,.$$

Next we observe that

- In the limit $R \to \infty$ the integral over C_R vanishes, as the integrand goes to zero as fast as $R^{-2/3}$.

- In the limit $\epsilon \to 0$ the integral over C_ϵ vanishes, as the integrand goes to zero as fast as $\epsilon^{4/3}$.

- In the limits $R \to \infty$ and $\epsilon \to 0$

$$\lim_{R\to\infty,\epsilon\to 0} \int_{R_+} \frac{\sqrt[3]{z}}{1+z^2} dz = \int_0^\infty \frac{\sqrt[3]{x}}{1+x^2} dx = I_2 \,,$$

$$\lim_{R\to\infty,\epsilon\to 0} \int_{R_-} \frac{\sqrt[3]{z}}{1+z^2} dz = -\int_0^\infty \frac{e^{i\frac{2}{3}\pi}\sqrt[3]{x}}{1+x^2} dx = \left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) I_2 \,.$$

Therefore, we have

$$J = \left(1 + \frac{1}{2} - i\frac{\sqrt{3}}{2}\right) I_2 = \frac{\sqrt{3}}{2}\left(\sqrt{3} - i\right) I_2 \,.$$

On the other hand, J is given by the sum of the residues at the simple poles $z = \pm i$ of the integrand

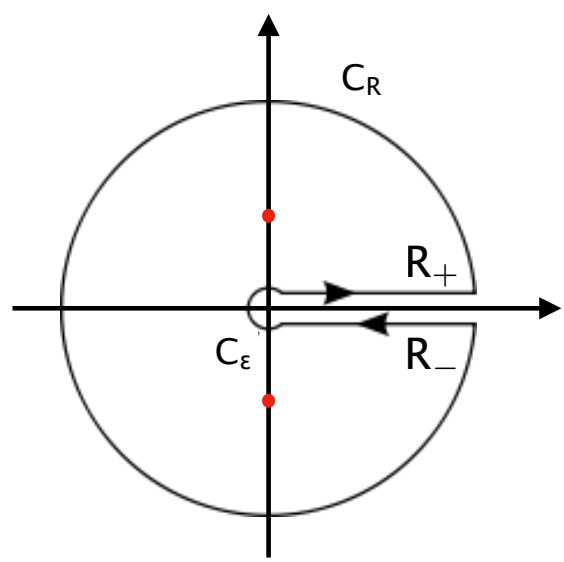
$$\begin{aligned} J =& 2\pi i \sum_{z_k=\pm i} \mathrm{Res} \frac{\sqrt[3]{z}}{1+z^2}\bigg|_{z=z_k} \\ =& \sum_{z_k=\pm i} \lim_{z\to z_k} \frac{\sqrt[3]{z}}{(z+i)(z-i)}(z-z_k) \\ =& 2\pi i \left(\frac{\sqrt[3]{i}}{2i} - \frac{\sqrt[3]{-i}}{2i}\right) \\ =& \frac{\pi}{2}\left(\sqrt{3} - i\right) \,. \end{aligned}$$

By combining the two results, we finally obtain

$$\frac{\sqrt{3}}{2}\left(\sqrt{3} - i\right) I_2 = \frac{\pi}{2}\left(\sqrt{3} - i\right) \,,$$

that gives

$$I_2 = \frac{\pi}{\sqrt{3}} \,.$$



Exercise 4

a) To begin with, we cast the differential equation (DE) into standard form

$$u'' + p(z)u' + q(z)u = 0\,,$$

with

$$p(z) = \frac{2z}{z^2-1}\,, \qquad q(z) = -\frac{3}{4(z^2-1)}\,.$$

We see immediately that $z = \pm 1$ are regular singular points of the DE, since they are simple poles of both $p(z)$ and $q(z)$. In order to study the behaviour of the DE at $z = \infty$ we perform the substitution

$$z = \frac{1}{t}\,, \qquad \frac{d}{dz} = -t^2\frac{d}{dt}\,, \qquad \frac{d^2}{dz^2} = t^4\frac{d^2}{dt^2} - 2t^3\frac{d}{dt}\,,$$

from which we get the DE in t

$$u'' + \tilde{p}(t)u' + \tilde{q}(t)u = 0\,,$$

with

$$\begin{aligned}\tilde{p}(t) =& \frac{2}{t} - \frac{1}{t^2}p\left(\frac{1}{t}\right) = \frac{2t}{t^2-1}\,,\\ \tilde{q}(t) =& \frac{1}{t^4}q\left(\frac{1}{t}\right) = \frac{3}{4t^2(t^2-1)}\,.\end{aligned}$$

Since $\tilde{p}(t)$ is regular in $t = 0$ and $\tilde{q}(t)$ has a double pole, we conclude that $z = \infty$ is another regular singular point of the DE. This means that, having only three regular singular points, the DE is fuchsian.

b) The characteristic exponents $\rho_{1,2}$ of the solution around a (finite) regular singular point z_0 of a fuchsian DE are found as roots of the characteristic equation

$$\rho^2 + (p_0 - 1)\rho + q_0 = 0\,,$$

with

$$p_0 = \lim_{z\to z_0}(z-z_0)p(z)\,, \qquad q_0 = \lim_{z\to z_0}(z-z_0)^2 q(z)\,.$$

For both $z_0 = \pm 1$ we have

$$\begin{aligned}p_0 &= \lim_{z\to\pm 1}(z\mp 1)\frac{2z}{(z+1)(z-1)} = \lim_{z\to\pm 1}\frac{2z}{(z\pm 1)} = 1\\ q_0 &= \lim_{z\to\pm 1}(z\mp 1)^2\frac{-3}{4(z^2-1)} = 0\,.\end{aligned}$$

Hence, the solutions around $z_0 = \pm 1$ have degenerate characteristic exponents given by $\rho^2 = 0$,

$$\rho_{1,2} = 0\,.$$

For $z = \infty$, we can study the characteristic equation in t

$$p_0 = \lim_{t\to 0} \frac{2t^2}{t^2-1} = 0\,, \qquad q_0 = \lim_{t\to 0} \frac{3t^2}{4t^2(t^2-1)} = -\frac{3}{4}\,.$$

Hence, we have $\rho^2 - \rho - 3/4 = 0$, which leads to

$$\rho 1 = \frac{3}{2}\,, \quad \rho_2 = -\frac{1}{2}\,.$$

From the knowledge of the characteristic exponents in all three regular singular points, we can immediately write down the formal general solution of the DE in terms of the Riemann symbol

$$u(z) = \left\{ \begin{matrix} -1 & 1 & \infty & \\ 0 & 0 & \frac{3}{2} & z \\ 0 & 0 & -\frac{1}{2} & \end{matrix} \right\}\,.$$

c) $z = 0$ is a regular point of the DE. Therefore, we can look for a series solution of the form

$$\begin{aligned} u(z) &= \sum_{k=0}^{\infty} c_k z^k\,, \\ u'(z) &= \sum_{k=0}^{\infty} c_k k z^{k-1}\,, \\ u''(z) &= \sum_{k=0}^{\infty} c_k k(k-1) z^{k-2}\,. \end{aligned}$$

By plugging this Ansatz directly into the DE, we obtain

$$\begin{aligned} &\sum_{k=2}^{\infty} c_k k(k-1) z^{k-1} + \sum_{k=0}^{\infty} \left(-k(k-1) - 2k + \frac{3}{4} \right) z^k \\ &= \sum_{k=0}^{\infty} \left[c_{k+2}(k+2)(k+1) - c_k \left(k^2 + k - \frac{3}{4} \right) \right] z^k \\ &= \sum_{k=0}^{\infty} \left[c_{k+2}(k+2)(k+1) - c_k \left(k + \frac{3}{2} \right) \left(k - \frac{1}{2} \right) \right] z^k = 0 \end{aligned}$$

By demanding the vanishing of the coefficient of each individual power of z, we get the recurrence relation

$$c_{k+2} = \frac{\left(k+\frac{3}{2}\right)\left(k-\frac{1}{2}\right)}{(k+2)(k+1)} c_k\,,$$

which leaves only the coefficients c_0 and c_1 undetermined. We can specialize this general solution to u_1 and u_2 by fixing the first two coefficients of the series. For u_1, we choose $c_0 = 1$ and $c_1 = 0$ and, from the recurrence relation, we have

$$c_2 = -\frac{3}{8}\,, \qquad c_4 = -\frac{21}{128}\,,$$

while all odd coefficients are zero. Conversely, for u_2, we set $c_0 = 0$ and $c_1 = 1$, so to get

$$c_3 = \frac{5}{64}\,, \qquad c_5 = -\frac{15}{128}\,,$$

and all even coefficients are zero. The requested solutions of the DE are then

$$\begin{aligned} u_1(z) =& 1 - \frac{3}{8}z^2 - \frac{21}{128}z^4 + \mathcal{O}(z^6)\,, \\ u_2(z) =& z + \frac{5}{24}z^3 - \frac{15}{128}z^5 + \mathcal{O}(z^7)\,. \end{aligned}$$

The two series converge for $|z| < 1$, as dictated by the distance from the closest singularities, $z = \pm 1$.

d) Let's have a closer look to the recurrence relation for the even coefficients. By setting $k = 2n$, we have

$$c_{2(n+1)} = \frac{\left(n+\frac{3}{4}\right)\left(n-\frac{1}{4}\right)}{(n+1)\left(n+\frac{1}{2}\right)} c_{2n}$$

If we iterate n-times this relation, we can express $c_{2(n+1)}$ in terms of c_0

$$\begin{aligned} c_{2(n+1)} =& \frac{\left(n+\frac{3}{4}\right)\left(n-\frac{1}{4}\right)\ldots\frac{3}{4}\left(n-\frac{1}{4}\right)\left(n-\frac{5}{4}\right)\left(-\frac{1}{4}\right)}{(n+1)\,n\ldots 1\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)\ldots\frac{1}{2}} c_0 \\ =& \frac{\left(\frac{3}{4}\right)_{n+1}\left(-\frac{1}{4}\right)_{n+1}}{(n+1)!\left(\frac{1}{2}\right)_{n+1}} c_0\,, \end{aligned}$$

or, shifting $n+1 \to n$

$$c_{2n} = \frac{\left(\frac{3}{4}\right)_n\left(-\frac{1}{4}\right)_n}{n!\left(\frac{1}{2}\right)_n} c_0\,.$$

Thus, u_1 can be rewritten as

$$u_1(z) = \sum_{n=0}^{\infty} c_{2n} z^{2n} = \sum_{n=0}^{\infty} \frac{\left(\frac{3}{4}\right)_n\left(-\frac{1}{4}\right)_n}{n!\left(\frac{1}{2}\right)_n}(z^2)^n = {}_2F_1\left(\frac{3}{4}, -\frac{1}{4}; \frac{1}{2}; z^2\right)\,.$$

For the odd coefficients, we set $k = 2n+1$ and get

$$c_{2(n+1)+1} = \frac{\left(n+\frac{5}{4}\right)\left(n+\frac{1}{4}\right)}{(n+1)\left(n+\frac{3}{2}\right)} c_{2n+1}\,.$$

After n steps,

$$\begin{aligned} c_{2(n+1)+1} =& \frac{\left(n+\frac{5}{4}\right)\left(n+\frac{1}{4}\right)\ldots\frac{5}{4}\left(n+\frac{1}{4}\right)\left(n-\frac{3}{4}\right)\frac{1}{4}}{(n+1)\,n\ldots 1\left(n+\frac{3}{2}\right)\left(n+\frac{1}{2}\right)\ldots\frac{3}{2}} c_1 \\ =& \frac{\left(\frac{5}{4}\right)_{n+1}\left(\frac{1}{4}\right)_{n+1}}{(n+1)!\left(\frac{3}{2}\right)_{n+1}} c_0\,, \end{aligned}$$

or, shifting $n+1 \to n$,

$$c_{2n+1} = \frac{\left(\frac{5}{4}\right)_n\left(\frac{1}{4}\right)_n}{n!\left(\frac{3}{2}\right)_n} c_1\,.$$

Thus, u_2 can be rewritten as

$$u_2(z) = \sum_{n=0}^{\infty} c_{2n+1} z^{2n+1} = z\sum_{n=0}^{\infty} \frac{\left(\frac{5}{4}\right)_n\left(\frac{1}{4}\right)_n}{n!\left(\frac{3}{2}\right)_n}(z^2)^n = z\,{}_2F_1\left(\frac{5}{4},\frac{1}{4};\frac{3}{2};z^2\right)\,.$$

Exercise 5

a) $\tilde{H} := \{hg^{-1} | g \in G, h \in H\} = HG = G$
check: $g \overset{?}{\in} \tilde{H}\ \forall g \in G \to e \in H$ thus g can be written in the form $g = eg \in \tilde{H}$
$gGg^{-1} = G\ \forall g \in G \Rightarrow \tilde{H} \triangleleft G$ since every group is a normal subgroup of itself.

b) We multiply i) with g^{-1} from the left to get to ii):

$$\begin{aligned} Hg &= gH \quad /g^{-1}\cdot \\ g^{-1}Hg &= g^{-1}gH = H \subset H \end{aligned}$$

$$\text{assume } gHg^{-1} \subsetneq H, \text{ and let } h \in H \text{ s.t. } h \notin gHg^{-1}: \quad ghg^{-1} = \tilde{h} \in gHg^{-1}$$
$$\Rightarrow \quad h = g^{-1}ghg^{-1}g = g^{-1}\tilde{h}g = \tilde{g}\tilde{h}\tilde{g}^{-1}, \quad \tilde{g} = g^{-1} \quad \Rightarrow \quad h \in gHg^{-1} \quad \lightning$$

And thus $gHg^{-1} \subset H \Rightarrow gHg^{-1} = H$. $\square$

c)

i) $\operatorname{sign}(\sigma)\operatorname{sign}(\pi)\operatorname{sign}(\sigma^{-1}) = \operatorname{sign}(\sigma)^2\operatorname{sign}(\pi) = \operatorname{sign}(\pi) \quad \forall \sigma, \pi \in S_3$,
$H_g = \{\pi \in S_3 | \operatorname{sign}(\pi) = 1\}$, $\quad H_u = \{\pi \in S_3 | \operatorname{sign}(\pi) = -1\} \quad \Rightarrow \quad$ both sets H_g and H_u are invariant under conjugation. Thus H_g is a normal subgroup, while H_u is not, since $id \notin H_u$.

ii) $S_3/H_g = \{gH_g | g \in S_3\}$, $id_{S_3/H_g} = idH_g = H_g$. Let $g \in H_g$ and $u, u' \in H_u$: $\operatorname{sign}(g)\operatorname{sign}(u) = -1 \quad \Rightarrow \quad gu \in H_u$ and $\operatorname{sign}(u)\operatorname{sign}(u') = 1 \quad \Rightarrow \quad uu' \in H_g$

S_3/H_g	H_g	H_u
H_g	H_g	H_u
H_u	H_u	H_g

d) $G/H = \{[g] | g \in G\}, \quad [g] = \{gh | h \in H\} = gH \quad \Rightarrow \quad |[g]| = |H| \quad \forall g \in G$

H partitions G: $\quad G = \bigcup_{g \in G} [g]$ and either $[g] = [g']$ or $[g] \bigcap [g'] = \emptyset \quad \Rightarrow \quad |G| = n|H|$,

where n is the number of different equivalence classes $[g]$ and thus $n = |G/H|$.

$$\Rightarrow \quad |G| = |G/H||H| \quad \Leftrightarrow \quad |G/H| = \frac{|G|}{|H|} \quad \square \qquad \text{1 pt}$$

e) The dihedral group D_n is the symmetry group of a regular n-gon i.e. operations on the n-gon that map it back to itself. Thus it comes with a natural group action on the set of points X of the n-gon:

$$\begin{array}{rccc} \cdot : & D_n \times X & \to & X \\ & (g, \vec{x}) & \mapsto & g \cdot \vec{x} \end{array}$$

Similarly S_n is defined as the group of bijections of the set $A = \{1, ..., n\}$ and thus also comes with a natural group action on A.
D_n can be generated by: $D_n = \langle R, S \rangle = \{\mathbf{1}, R, .., R^{n-1}, S, .., R^{n-1}S\}$. Let us restrict the group action of D_n to the n-tuple of edge points $\vec{x}^* := (\vec{x}_1, ..., \vec{x}_n)$, where $\vec{x}_i$ is the position of an edge on the plane E (using $E \simeq \mathbb{R}^2$).

$$R\vec{x}^* = (\vec{x}_n, \vec{x}_1, .., \vec{x}_{n-1}), \quad S\vec{x}^* = \begin{cases} (\vec{x}_n, \vec{x}_{n-1}, ..., \vec{x}_2, \vec{x}_1), & n \text{ even} \\ (\vec{x}_{n-1}, \vec{x}_{n-2}, ..., \vec{x}_2, \vec{x}_1, \vec{x}_n), & n \text{ odd} \end{cases}$$

We could achieve the same, by permuting the indices with $\sigma \in S_n$ instead:

$$\begin{array}{rccc} * : & S_n \times E^n & \to & E^n \\ & (\sigma, \vec{x}) & \mapsto & \sigma * \vec{x} \end{array}, \qquad \sigma * \vec{x} = (\vec{x}_{\sigma(i)})_{i=1}^n$$

Using the two group actions, we can map the generators of D_n to S_n and since they are mapped by group actions, the group operations are preserved:

$$g \in D_n \overset{\cdot}{\mapsto} (g\cdot : E^n \to E^n) = (\sigma * : E^n \to E^n) \overset{*}{\mapsto} \sigma \in S_n$$

$$R \mapsto \sigma_R := (12...n), \quad S \mapsto \sigma_S := \begin{cases} (1n)(2[n-1])...(\frac{n}{2}\frac{n+2}{2}) & n \text{ even} \\ (1[n-1])(2[n-2])...(\frac{n-1}{2}\frac{n+1}{2}), & n \text{ odd} \end{cases}$$

Thus $\langle \sigma_R, \sigma_S \rangle \simeq D_n$ and since $|D_n| = 2n$ and $|S_n| = n!$: $\langle \sigma_R, \sigma_S \rangle = S_n$ for $n = 3$ and $\langle \sigma_R, \sigma_S \rangle < S_n$ for $n > 3$. $\square$

f) Let G be a group, V a vector space and $\rho : G \to GL(V)$ a representation. Define:

$$\begin{array}{rccc} \cdot : & G \times V & \to & V \\ & (g, v) & \mapsto & \rho(g)v \end{array}$$

$$g \cdot (g' \cdot v) = \rho(g)(\rho(g')v) = \rho(gg')v = (gg') \cdot v \quad \square$$

Checking the group map property of the group action shows, that we have just written the representation in a different way.
A group action only requires a set and the action of the group on that set does not have to be linear. So, a representation is a special case of a group action. 1 pt

Exercise 6

a) i)

$$\begin{aligned} \alpha_1\Omega_1 + \alpha_2\Omega_2 =& \alpha_1 \frac{d}{dt} R_1(t)|_{t=0} + \alpha_2 \frac{d}{dt} R_2(t)|_{t=0} && \Big| \text{ using chain rule} \\ =& \frac{d}{dt} R_1(\alpha_1 t)|_{t=0} + \frac{d}{dt} R_2(\alpha_2 t)|_{t=0} && \Big| \text{ using product rule} \\ =& \frac{d}{dt} R_1(\alpha_1 t) R_2(\alpha_2 t)|_{t=0} \end{aligned}$$

$R_1(\alpha_1 t)R_2(\alpha_2 t) \in SO(n)$ since $\alpha\mathbb{R} = \mathbb{R}$ and $R_1R_2 \in SO(n)$ $\forall R_1, R_2 \in SO(n)$ and $R_1(\alpha_1 t)R_2(\alpha_2 t)|_{t=0} = R_1(0)R_2(0) = \mathbb{I}^2 = \mathbb{I}$

ii) $SO(n) = \{R \in \mathbb{R}^{n\times n} | R^T R = \mathbb{I}, \det R = 1\}$

$$\frac{d}{dt} R^T(t)R(t)|_{t=0} = \frac{d}{dt} R^T(t)|_{t=0} + \frac{d}{dt} R(t)|_{t=0} = \Omega^T + \Omega = \frac{d}{dt}\mathbb{I}|_{t=0} = 0 \quad \Rightarrow \quad \Omega^T = -\Omega \quad \square$$

b) The group multiplication becomes a vector addition in the Lie-algebra.

$$\begin{aligned} \tilde{\rho}(\Omega)^\dagger &= \left(\frac{d}{dt} \rho(R(t))|_{t=0} \right)^\dagger = \frac{d}{dt} \rho(R(t))^\dagger|_{t=0} = \frac{d}{dt} \rho(R^{-1}(t))|_{t=0} \\ &= \frac{d}{dt} \rho(R(-t))|_{t=0} = -\frac{d}{dt} \rho(R(t))|_{t=0} = -\tilde{\rho}(\Omega) \qquad \square \end{aligned}$$

c) Written in components we can express skew symmetry as:

$$\Omega_{ij} = -\Omega_{ji} \quad \Rightarrow \quad \Omega_{ii} = -\Omega_{ii} \quad \Rightarrow \quad \Omega_{ii} = 0 \quad \text{for the diagonal.}$$

So, the dimension of the space of all skew-symmetric $n \times n$-matrices is $\frac{n(n-1)}{2}$. For Ω to be in $\mathfrak{so}(n)$ it is also requires to be trace less, since $\det R = 1 \; \forall R \in SO(n)$ translates into:

$$0 = \frac{d}{dt} 1|_{t=0} = \frac{d}{dt} \det R(t)|_{t=0} = \det(R(0)) \operatorname{Tr}\left(A^{-1}(t) \frac{d}{dt} A(t) \right)\Big|_{t=0} = \operatorname{Tr}(\mathbb{I}\Omega) = \operatorname{Tr}\Omega$$

However, this condition is already satisfied, since $\Omega_{ii} = 0$ and thus does not further constrain the degrees of freedom. Finally the dimension of the Lie-algebra is $\dim \mathfrak{so}(n) = \frac{n(n-1)}{2}$ and since the dimensions of a manifold and its tangent spaces always agree: $\underline{\underline{\dim SO(n) = \frac{n(n-1)}{2}}}$

Exercise 7

a) i) $\rho(id) = \mathbb{1}, \quad \rho(13) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \rho(132) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

ii) No, $\langle (1,1,1) \rangle$ is an invariant subspace.

iii)

$$\rho = \bigoplus_i m_i \rho_i, \quad m_i = (\chi_\rho, \chi_i) = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_i(g)} \chi_\rho(g) = \frac{1}{|G|} \sum_K |K| \overline{\chi_i(K)} \chi_\rho(K),$$

where ρ_i are irreducible representations and K are the conjugacy classes of G.

$$\chi_\rho([id]) = 3, \quad \chi_\rho([(13)]) = 1, \quad \chi_\rho([(132)]) = 0$$

S_3	id	$2(123)$	$3(12)$
trivial: t	1	1	1
alternating: a	1	1	-1
standard: std	2	-1	0

$\xrightarrow{\sim}$

C_{3v}	E	$2C_3$	$3\sigma_v^{(1)}$
A_1	1	1	1
A_2	1	1	-1
E	2	-1	0

$$\left.\begin{aligned} m_t &= \tfrac{1}{6}(1 \cdot 1 \cdot 3 + 2 \cdot 1 \cdot 0 + 3 \cdot 1 \cdot 1) = 1 \\ m_a &= \tfrac{1}{6}(1 \cdot 1 \cdot 3 + 2 \cdot 1 \cdot 0 + 3 \cdot (-1) \cdot 1) = 0 \\ m_{std} &= \tfrac{1}{6}(1 \cdot 2 \cdot 3 + 2 \cdot (-1) \cdot 0 + 3 \cdot 0 \cdot 1) = 1 \end{aligned}\right\} \quad \Rightarrow \quad \rho = \rho_t \oplus \rho_{std}$$

b) Following the steps in the script p.68, "Charakter Tafel einer endlichen Gruppe":

1) number of irreducible representations: $n_I = n_K$
$[\pi] = \{\sigma \in H | \exists h \in H : h\sigma h^{-1} = \pi\}$, let $\sigma_+ = (123)$ and $\sigma_- = (321)$:

$$\sigma_+\sigma_- = (123)(321) = (1)(2)(3) = id \quad \Rightarrow \quad \sigma_+^{-1} = \sigma_-$$
$$\sigma_+^2 = (123)(123) = (132) = \sigma_-, \quad \sigma_-^2 = (321)(321) = (123) = \sigma_+$$
$$\Rightarrow \left.\begin{array}{l} \sigma_-\sigma_+\sigma_-^{-1} = \sigma_-\sigma_+^2 = \sigma_-^2 = \sigma_+ \\ \sigma_+\sigma_+\sigma_+^{-1} = \sigma_+^2\sigma_- = \sigma_-^2 = \sigma_+ \end{array}\right\} \Rightarrow \left.\begin{array}{l} [\sigma_+] = \sigma_+ \\ [\sigma_-] = \sigma_- \end{array}\right\} \Rightarrow \quad n_K = 3$$

2) $|G| = \sum_{\nu=1}^{n_I} d_\nu^2$:
$3 = 1^2 + d_2^2 + d_3^2 \quad \Rightarrow \quad d_2 = d_3 = 1$

3) $\chi_\nu(id) = d_\nu$

4) $\chi_t(\pi) = 1 \ \forall \pi \in H$

$\Rightarrow$

H	id	(123)	(321)
t	1	1	1
ρ_2	1	$x = e^{i\phi}$	$x' = e^{i\phi'}$
ρ_3	1	$y = e^{i\theta}$	$y' = e^{i\theta'}$

5) $\sum_{K=1}^{n_K} g_K \chi_\nu(K)\overline{\chi_\gamma(K)} = |G|\delta_{\nu\gamma}$

$$t, \rho_2: \quad 1 + e^{-i\phi} + e^{-i\phi'} = 0 \qquad \text{(I)}, \qquad t, \rho_3: \quad 1 + e^{-i\theta} + e^{-i\theta'} = 0 \quad \text{(II)}$$
$$\rho_2, \rho_3: \quad 1 + e^{i(\phi-\theta)} + e^{i(\phi'-\theta')} = 0 \qquad \text{(III)}$$

The equations from ρ_2, ρ_2 and ρ_3, ρ_3 are trivial: $1 + e^0 + e^0 = 0$.

6) $\sum_{\nu=1}^{n} \chi_\nu(K)\overline{\chi_\nu(K')} = \frac{|G|}{g_K}\delta_{KK'}$

$$id, (123): \quad 1 + e^{-i\phi} + e^{-i\theta} = 0 \qquad \text{(IV)}, \qquad id, (321): \quad 1 + e^{-i\phi'} + e^{-i\theta'} = 0 \quad \text{(V)}$$
$$(123), (321): \quad 1 + e^{i(\phi-\phi')} + e^{i(\theta-\theta')} = 0 \qquad \text{(VI)}$$

The equation for $(123), (123)$ and $(321), (321)$ is the same trivial one as above.

7) $g_{K_i} g_{K_j} \chi_\nu(K_i)\chi_\nu(K_j) = d_\nu \sum_l c_{ijl} g_{K_l} \chi_\nu(K_l), \quad K_i \times K_j = \sum_K c_{ijl} K_l$

$$[id] \times [\sigma_\pm] = id \circ \sigma_\pm \quad \Rightarrow \quad c_{id\ \sigma_\pm l} = \delta_{l\sigma_\pm}: \qquad \chi_i(\sigma_\pm) = \chi_i(\sigma_\pm) \qquad \text{trivial equation}$$
$$[\sigma_+] \times [\sigma_-] = [id] \quad \Rightarrow \quad c_{\sigma_+\sigma_- l} = \delta_{l\ id}: \quad \chi_\nu(\sigma_+)\chi_\nu(\sigma_-) = \chi_\nu(id)$$
$$\Rightarrow \quad \nu = \begin{cases} 2: & e^{i\phi}e^{i\phi'} = 1 \\ 3: & e^{i\theta}e^{i\theta'} = 1 \end{cases} \qquad (*)$$
$$[\sigma_\pm] \times [\sigma_\pm] = [\sigma_\mp] \quad \Rightarrow \quad c_{\sigma_\pm\sigma_\pm l} = \delta_{l\sigma_\mp}: \quad \chi_\nu^2(\sigma_\pm) = \chi(\sigma_\mp)$$
$$\Rightarrow \quad \nu = \begin{cases} 2: & e^{2i\phi} = e^{i\phi'}, \quad e^{2i\phi'} = e^{i\phi} \\ 3: & e^{2i\theta} = e^{i\theta'}, \quad e^{2i\theta'} = e^{i\theta} \end{cases} \qquad (**)$$

Using the equations 5)-7) we can determine ϕ, ϕ', θ and θ'.

$$\begin{aligned}
(\mathrm{I}), (\mathrm{IV}): \quad & e^{-i\phi} = e^{-i(\phi'+\pi)} = e^{-i(\theta+\pi)} \quad \Rightarrow \quad \theta = \phi' \\
(\mathrm{II}), (\mathrm{IV}): \quad & e^{-i\theta} = e^{-i(\theta'+\pi)} = e^{-i(\phi+\pi)} \quad \Rightarrow \quad \theta' = \phi
\end{aligned}$$

Thus, (III) and (V) are equivalent and we can rewrite the equation to get:

$$1 + e^{i(\phi-\phi')} + e^{i(\phi'-\phi)} = 1 + 2\cos(\phi - \phi') = 0 \quad \Rightarrow \quad \cos(\phi - \phi') = -\frac{1}{2} \quad \Rightarrow \quad \phi - \phi' = \pm\frac{2\pi}{3}$$

$$(**): \quad e^{i(2\phi-\phi')} = e^{i(\phi\pm\frac{2\pi}{3})} = 1 \quad \Rightarrow \quad \phi = \mp\frac{2\pi}{3} \text{ and } \phi' = \mp\frac{4\pi}{3}$$

Check for consistency: $(*): \quad e^{i(\phi+\phi')} = e^{\pm i\frac{6\pi}{3}} = 1$ and $(**): \quad e^{2i\phi'} = e^{\pm i\frac{8\pi}{3}} = e^{\pm i\frac{2\pi}{3}} = e^{i\phi}$

And we finally get the character table:

H	id	(123)	(321)
t	1	1	1
ρ_2	1	$x = e^{\pm i\frac{2\pi}{3}}$	$x' = e^{\pm i\frac{4\pi}{3}}$
ρ_3	1	$y = e^{\pm i\frac{4\pi}{3}}$	$y' = e^{i\pm i\frac{2\pi}{3}}$

Exercise 8

a) i) R_{ijkm} is a tensor of rank 4 in 4 dimensions and therefore has $4 \times 4 \times 4 \times 4 = 256$ independent components in general.

ii) R_{ijkm} has anti-symmetry in the first two indices and the last two indices (skew symmetry). To satisfy this, all components of the type R_{iiii}, $R_{iijj}(i \neq j)$, $R_{iikm}(i \neq k \neq m)$, $R_{ijkk}(i \neq j \neq k)$ and all permutations of $R_{iiij}(i \neq j)$ have to be zero. Only 36 components remain.

iii) R_{ijkm} is symmetric under exchange of its first and last pair of indices (interchange symmetry), which leaves us with 21 independent components. (You can show this by arranging the remaining components from ii) as a 6×6 matrix with indices $\{ij\}$ and $\{km\}$. Under interchange symmetry, only the diagonal and half of the off-diagonal elements remain independent.)

b) Covariant transformation $g_{ab} \to \overline{g}_{ij}$ with the appropriate partial derivatives:

$$g_{ab} = \overline{g}_{ij} \frac{\partial \overline{x}^i}{\partial x^a} \frac{\partial \overline{x}^j}{\partial x^b}$$