MMP 2
Solutions 12

FS 19
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Exercise 1

$$\mathrm{U(n)} = \{U \in \mathbb{C}^{n\times n} | U^\dagger U = \mathbb{1}\}, \quad \mathrm{O(n)} = \{A \in \mathbb{R}^{n\times n} | A^T A = \mathbb{1}\}$$

$1_{\mathrm{O(n)}} = \mathbb{1} = 1_{\mathrm{U(n)}}, \quad \forall A \in \mathrm{O(n)}: \quad A^\dagger A = \overline{A}^T A = A^T A = \mathbb{1} \quad \Rightarrow \quad \emptyset \neq \mathrm{O(n)} \subseteq \mathrm{U(n)}$

O(n) is itself a group:

$$\forall A, B \in \mathrm{O(n)}: \quad (AB)^T AB = B^T A^T AB = B^T \mathbb{1} B = \mathbb{1} \qquad \Rightarrow \quad AB \in \mathrm{O(n)}$$
$$\forall A \in \mathrm{O(n)}: \quad A^{-1} = A^T, \quad (A^T)^T A^T = AA^T = AA^{-1} = \mathbb{1} \qquad \Rightarrow \quad A^{-1} \in \mathrm{O(n)}$$ 3 pt

- $\mathrm{O(n)} \ntriangleleft \mathrm{U(n)}$; counter example: $U \in \mathrm{U}(2)$, $A \in \mathrm{O}(2)$, $\quad UAU^{-1} =$

$$= \begin{pmatrix} -i & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} i\cos\varphi & -\sin\varphi \\ i\sin\varphi & \cos\varphi \end{pmatrix} = \begin{pmatrix} \cos\varphi & i\sin\varphi \\ i\sin\varphi & \cos\varphi \end{pmatrix} \notin \mathrm{O}(2)$$

- $\mathrm{SO(n)} \triangleleft \mathrm{O(n)} \leq \mathrm{U(n)}$ is correct:
  SO(n) is a subgroup of O(n) and since: $\det(ARA^{-1}) = \det(A)\det(R)\det(A^{-1}) = \det(R) = 1$
  $\forall A \in \mathrm{O(n)}$, $\forall R \in \mathrm{SO(n)}$, SO(n) is normal and $\mathrm{O(n)} \leq \mathrm{U(n)}$ was proven above.

- $\mathrm{O(n)} \nleq \mathrm{SU(n)}$:
  $\forall A \in \mathrm{O(n)}$: $A^T A = \mathbb{1} \Rightarrow \det(A^T A) = \det(A^T)\det(A) = \det(A)^2 = \det(\mathbb{1}) = 1 \Rightarrow \det(A) = \pm 1$
  For example let $A = \begin{pmatrix} -1 & 0 \\ 0 & \mathbb{1}_{n-1} \end{pmatrix}$: $\quad A^T A = A^2 = \mathbb{1} \quad \Rightarrow \quad A \in \mathrm{O(n)}$,
  but $\det(A) = -1 \cdot 1^{n-1} = -1 \quad \Rightarrow \quad A \notin \mathrm{SU(n)}$ 1 pt

Unless $n = 1$, none of these matrix groups is abelian. 1 pt

Exercise 2

$$Lie(G) := \{X \in \mathbb{K}^{n\times n} | \exp(tX) \in G \quad \forall t \in \mathbb{R}\}$$

For any matrix Lie group G, $\dot{A}(0) = \frac{d}{dt}A(t)\big|_{t=0} \in \mathbb{K}^{n\times n}$ and $A(t) = \exp(t\dot{A}(0)) \in G \ \forall t \in \mathbb{R} \Rightarrow T_eG \subseteq Lie(G)$. On the other hand $\exp(tX)$ defines a curve on G through $e = \mathbb{1}$, since $\exp(O) = \mathbb{1}$, where O denotes the zero matrix, and $\frac{d}{dt}\exp(tX)\big|_{t=0} = X$, thus $Lie(G) \subseteq T_eG$.
$\Rightarrow \quad T_eG = Lie(G)$ 1 pt

$$ad(X)(Y) = \frac{d}{dt}\mathrm{Ad}(A(t))(Y)\bigg|_{t=0} = \frac{d}{dt}A(t)YA^{-1}(t)\bigg|_{t=0} = \dot{A}(0)YA^{-1}(0) + A(0)Y\dot{A}^{-1}(0) = XY - YX = [X,Y],$$

since $\dot{A}(0) = X, A(0) = \mathbb{1}$

and thus $A^{-1}(0) = A(0)^{-1} = \mathbb{1}, \quad \dot{A}^{-1}(0) = \frac{d}{dt}\exp(tX)^{-1}\bigg|_{t=0} = \frac{d}{dt}\exp(-tX)\bigg|_{t=0} = -X \qquad \square$ 2 pt

The boosts and spacial rotations trace out the degrees of freedom of the Lorentz group and thus their Lie algebra elements form a basis of its Lie algebra.

$$K_z = \dot{B}_z(0) = \frac{d}{d\lambda}\begin{pmatrix} \cosh\lambda & 0 & 0 & \sinh\lambda \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\lambda & 0 & 0 & \cosh\lambda \end{pmatrix}\bigg|_{\lambda=0} = \begin{pmatrix} \sinh(0) & 0 & 0 & \cosh(0) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \cosh(0) & 0 & 0 & \sinh(0) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\text{analogue} \quad K_x = \dot{B}_x(0) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad K_y = \dot{B}_y(0) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$L_z = \dot{R}_z(0) = \frac{d}{d\theta}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\bigg|_{t=0} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\sin(0) & -\cos(0) & 0 \\ 0 & \cos(0) & -\sin(0) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{analogue} \quad L_x = \dot{R}_x(0) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad L_y = \dot{R}_y(0) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$ 2 pt

Exercise 3

An $\binom{r}{s}$-tensor is a map $T : V^r \to V^s$ and thus $T \in \mathcal{T}^r_s(\mathbb{R}^n) = (V^*)^{\otimes r} \otimes V^{\otimes s}$

$x^T = (x_i)_{i=1}^n$ is a $\binom{1}{0}$-tensor $\Leftrightarrow x^T \in V^*$ ($x^T : V \to \mathbb{R}$), i.e. a covariant tensor of rank 1.
$y = (y^i)_{i=1}^n$ is a $\binom{0}{1}$-tensor, $\Leftrightarrow y \in V$, i.e. a contravariant tensor of rank 1.
$M = (M^i_j)_{i,j=1}^n$ is a $\binom{1}{1}$-tensor, $\Leftrightarrow M \in V^* \otimes V$ ($M : V^* \to V$), i.e. a mixed tensor of rank 2. 1 pt

$$x \mapsto \phi(x) = x'; \quad g_x(X,Y) = g_{\mu\nu}X^\mu Y^\nu = g_{\mu\nu}(x)\frac{\partial x^\mu}{\partial x'^\rho}X'^\rho\frac{\partial x^\nu}{\partial x'^\sigma}Y'^\sigma = g'_{\rho\sigma}(x')X'^\rho Y'^\sigma = g'_{x'}(X',Y')$$
$$\Rightarrow \quad g'_{\rho\sigma}(\phi(x)) = \frac{\partial x^\mu}{\partial x'^\rho}\frac{\partial x^\nu}{\partial x'^\sigma}g_{\mu\nu}(x)$$

It transforms like a covariant tensor of rank 2 for all positions $x \in \mathbb{R}^n$ and therefore is a $\binom{2}{0}$-tensor field. $\square$

- $g'_{\mu\nu}(x') + h_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}g_{\rho\sigma}(x) + \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}h_{\rho\sigma}(x) = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\left[g_{\rho\sigma}(x) + h_{\rho\sigma}(x)\right]$
  $\Rightarrow \quad g_{\mu\nu}(x) + h_{\mu\nu}(x) \in \mathcal{T}^2_0(\mathbb{R}^n)$

- $\frac{\partial}{\partial x'^\mu}g'_{\rho\sigma}(x') = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu}\frac{\partial x^\lambda}{\partial x'^\rho}\frac{\partial x^\eta}{\partial x'^\sigma}g_{\lambda\eta}(x) \overset{i.g.}{\neq} \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial x^\lambda}{\partial x'^\rho}\frac{\partial x^\eta}{\partial x'^\sigma}\frac{\partial}{\partial x^\nu}g_{\lambda\eta}(x) \quad \Rightarrow \quad g_{\rho\sigma,\mu}(x) \notin \mathcal{T}^3_0(\mathbb{R}^n)$

- $g'^{\mu\eta}(x')g'_{\eta\nu}(x) = \frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial x'^\eta}{\partial x^\lambda}g^{\rho\lambda}(x)\frac{\partial x^\kappa}{\partial x'^\eta}\frac{\partial x^\sigma}{\partial x'^\nu}g_{\kappa\sigma}(x) = \frac{\partial x'^\mu}{\partial x^\rho}\underbrace{\frac{\partial x'^\eta}{\partial x^\lambda}\frac{\partial x^\kappa}{\partial x'^\eta}}_{\delta^\lambda_\kappa}\frac{\partial x^\sigma}{\partial x'^\nu}g^{\rho\lambda}(x)g_{\kappa\sigma}(x)$
  $= \frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial x^\sigma}{\partial x'^\nu}g^{\rho\lambda}(x)g_{\lambda\sigma}(x) \quad \Rightarrow \quad g^{\mu\eta}(x)g_{\eta\nu}(x) \in \mathcal{T}^1_1(\mathbb{R}^n)$