## Exercise 1. *Goldberger-Treiman relation*

*The matrix element of the axial isospin current in the nucleon can be written in terms of form factors as*

$$\langle N|J^{\mu 5a}(q)|N\rangle = \bar{u}(p')\left[\gamma^\mu\gamma^5 F_1^5(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\gamma^5 F_2^5(q^2) + q^\mu\gamma^5 F_3^5(q^2)\right]\tau^a u(p)\,, \tag{1}$$

*with $q = p - p'$ and $\tau^a = \sigma^a/2$.*

1. *In the limit of massless quarks, assume the conservation of the axial vector current and show that*

$$g_A := F_1^5(0) = \lim_{q^2\to 0}\frac{q^2}{2m_N}F_3^5(q^2)\,, \tag{2}$$

*where $m_N$ is the mass of the nucleon.*

2. *Given that the low-energy pion-nucleon interaction is parametrised by the Lagrangian*

$$\mathscr{L}_{int} = 2ig_{\pi NN}\pi^a\bar{N}\gamma^5\tau^a N\,, \tag{3}$$

*consider the contribution to the matrix element (1) where the interaction is mediated by the exchange of a massless pion. Show the Goldberger-Treiman relation*

$$g_A = \frac{f_\pi}{m_N}g_{\pi NN}. \tag{4}$$

## Solution.

1. If we demand the conservation of the axial vector current, $q_\mu J^{\mu 5a}(q) = 0$, on the matrix element of eq. (1), we have

$$\langle N|q_\mu J^{\mu 5a}(q)|N\rangle = \bar{u}(p')\left[\not{q}\gamma^5 F_1^5(q^2) + \frac{i\sigma^{\mu\nu}q_\nu q_\mu}{2m}\gamma^5 F_2^5(q^2) + q^2\gamma^5 F_3^5(q^2)\right]\tau^a u(p) = 0\,. \tag{S.1}$$

The term proportional to $\sigma^{\mu\nu}$ term vanishes by parity. In addition, we can use momentum conservation $q = p - p'$ on the first term and apply the equations of motion

$$\bar{u}(p')(\not{p}' - m_N) = 0\,, \qquad (\not{p} - m_N)u(p) = 0 \tag{S.2}$$

in order to arrive at

$$\bar{u}(p')\left[-2m_N\gamma^5 F_1^5(q^2) + q^2\gamma^5 F_3^5(q^2)\right]\tau^a u(p) = 0\,. \tag{S.3}$$

Note that we have obtained the factor $-2m_N$ due to the anticommutation of $\gamma^5$ with $\gamma^\mu$. In the $q^2 \to 0$ limit, eq. (S.3) becomes,

$$g_A = F_1^5(0) = \lim_{q^2\to 0}\frac{q^2 F_3^5(q^2)}{2m_N}. \tag{S.4}$$

2. As we can see from eq. (S.4), $g_A$ can only be non-zero if $F_3^5(q^2)$ has a simple pole in $q^2 = 0$. Such singularity corresponds to the exchange of a massless pion between the axial-vector current and the nucleon. The matrix element associated to the diagrams of Fig. 1 reads

$$\begin{aligned}\mathscr{M} &= (iq^\mu f_\pi)\left(\frac{i}{q^2}\right)\left(-2g_{\pi NN}\bar{u}(p')\gamma^5\tau^a u(p)\right)\\ &= \frac{2g_{\pi NN}f_\pi}{q^2}\bar{u}(p')\gamma^5\tau^a u(p)\,,\end{aligned} \tag{S.5}$$

where we have used eq. (3) as well as the expression of the matrix element $J^{\mu 5a}(q)$ between the vacuum and the pion state from Exercise Sheet 4. As expected, the pion exchange gives a contribution to the form factor $F_3^5(q^2)$,

$$F_3^5(q^2) \underset{q^2 \to 0}{\simeq} \frac{2 f_\pi g_{\pi NN}}{q^2}\,, \tag{S.6}$$

so that eq. (S.4) becomes

$$g_a = F_1^5(0) = \frac{f_\pi g_{\pi NN}}{m_N}. \tag{S.7}$$

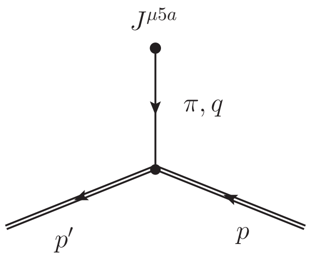


Figure 1: Scattering of an axial-vector current $J^{\mu 5a}$ on a nucleon via pion exchange.

**Exercise 2.** ***Gell-Mann–Okubo Mass Formula and Weinberg Ratio***

*Consider the Lagrangian of chiral perturbation theory at order* $p^2$,

$$\mathscr{L}_{CHPT,\,p^2} = \frac{v^2}{4}\,\mathrm{Tr}\left(D_\mu U D^\mu U^\dagger + \chi U^\dagger + \chi^\dagger U\right), \tag{5}$$

*where* $U = \exp(i\sqrt{2}\Phi/v)$ *is a* $SU(3)$ *matrix with*

$$\Phi = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -2\frac{\eta_8}{\sqrt{6}} \end{pmatrix}, \tag{6}$$

*and* $\chi = 2B\, diag\,(m_u, m_d, m_s)$ *is the quark mass matrix. Expand the Lagrangian in* $\Phi$ *and compute the mass of the eight mesons. Verify the Gell-Mann–Okubo mass formula*

$$4m_K^2 - 3m_\eta^2 - m_\pi^2 = 0 \tag{7}$$

*and the Weinberg ratio of quark masses*

$$\frac{2m_K^2 - m_\pi^2}{m_\pi^2} = \frac{2m_s}{m_d + m_u}\,, \tag{8}$$

*where* $m_\pi^2$ *and* $m_K^2$ *are the average masses of the pion and the kaon, respectively.*

**Solution.** In order to obtain the kinetic term of $\mathscr{L}_{\mathrm{CHPT},\,p^2}$, we can replace $D_\mu \to \partial_\mu$ and expand the field matrix $U$ up to the $\Phi^2$ term,

$$\begin{aligned} U &= 1 + i\frac{\sqrt{2}}{v}\Phi - \frac{2}{v^2}\Phi^2 + \mathscr{O}(\Phi^3)\,, \\ U^\dagger &= 1 - i\frac{\sqrt{2}}{v}\Phi - \frac{2}{v^2}\Phi^2 + \mathscr{O}(\Phi^3)\,. \end{aligned} \tag{S.8}$$

In the the derivate part, we retain the quadric term only. In addition, the reality of $\chi$ makes the linear terms cancel each other. Hence, we obtain (up to an constant term that we can omit)

$$\mathscr{L}_{\mathrm{CHPT},\,p^2} = \frac{1}{2}\,\mathrm{Tr}\,(\partial_\mu \Phi \partial^\mu \Phi) + \mathrm{Tr}\left(BM\Phi^2\right) + \ldots \tag{S.9}$$

By computing the two traces explicitly, we can write the above Lagrangian as a sum of Lagrangians for real ($\pi^0$, $\eta_8$) and complex ($\pi^\pm$, $K^0$, $K^\pm$) scalar fields plus a $\pi^0$-$\eta_8$ interaction term,

$$\mathscr{L}_{\mathrm{CHPT},\,p^2} = \mathscr{L}^{\mathrm{kin}}_{\pi^0} + \mathscr{L}^{\mathrm{kin}}_{\pi^\pm} + \mathscr{L}^{\mathrm{kin}}_{\eta_8} + \mathscr{L}^{\mathrm{kin}}_{K^0} + \mathscr{L}^{\mathrm{kin}}_{K^\pm} + \mathscr{L}^{\mathrm{int}}_{\pi^0\eta_8}\,, \tag{S.10}$$

with

$$\begin{aligned}
\mathscr{L}^{\mathrm{kin}}_{\pi^0} =& \frac{1}{2}\partial^\mu \pi^0 \partial_\mu \pi^0 - \frac{1}{2} m^2_{\pi^0} (\pi^0)^2 \\
\mathscr{L}^{\mathrm{kin}}_{\pi^\pm} =& \partial^\mu \pi^+ \partial_\mu \pi^- - m^2_{\pi^\pm} \pi^+ \pi^- \\
\mathscr{L}^{\mathrm{kin}}_{\eta_8} =& \frac{1}{2}\partial^\mu \eta_8 \partial_\mu \pi_8 - \frac{1}{2} m^2_{\eta_8} (\eta_8)^2 \\
\mathscr{L}^{\mathrm{kin}}_{K^0} =& \partial^\mu K^0 \partial_\mu \bar{K}^0 - \frac{1}{2} m^2_{K^0} \bar{K}^0 K^0 \\
\mathscr{L}^{\mathrm{kin}}_{K^\pm} =& \partial^\mu K^+ \partial_\mu K^- - m^2_{K^\pm} K^+ K^- \\
\mathscr{L}^{\mathrm{int}}_{\pi^0\eta_8} =& \frac{B}{\sqrt{3}} (m_d - m_u) \pi^0 \eta_8\,.
\end{aligned} \tag{S.11}$$

The masses of the mesons are given by

$$\begin{aligned}
m^2_{\pi^0} =& B(m_d + m_u)\,, \\
m^2_{\pi^+} =& B(m_u + m_d)\,, \\
m^2_{\eta_8} =& \frac{B}{3}\left(m_u + m_d + 4m_s\right)\,, \\
m^2_{K^0} =& B(m_d + m_s)\,, \\
m^2_{K^+} =& B(m_u + m_s)\,,
\end{aligned} \tag{S.12}$$

and it can be easily checked that they obey both the Gell-Mann–Okubo relation and the Weinberg ratio of quark masses.