Universität Zürich[UZH]

Physik-Institut

Kern- und Teilchenphysik II
Lecture 1: QCD

(adapted from the Handout of Prof. Mark Thomson)

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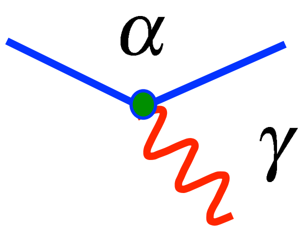
QCD

★ **The theory of the strong interaction, Quantum Chromodynamics (QCD), is very similar to QED but with 3 conserved "colour" charges**

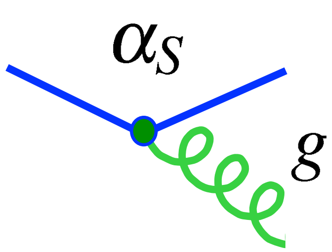
<u>In QED:</u>

- **the electron carries one unit of charge** $-e$
- **the anti-electron carries one unit of anti-charge** $+e$
- **the force is mediated by a massless "gauge boson" – the photon**



<u>In QCD:</u>

- **quarks carry colour charge:** r, g, b
- **anti-quarks carry anti-charge:** $\overline{r}, \overline{g}, \overline{b}$
- **The force is mediated by massless gluons**



★ **In QCD, the strong interaction is invariant under rotations in colour space**

$$r \leftrightarrow b;\; r \leftrightarrow g;\; b \leftrightarrow g$$

i.e. the same for all three colours

⇨ $\boxed{\textbf{SU(3) colour symmetry}}$

•**This is an exact symmetry, unlike the approximate uds flavour symmetry discussed previously.**

Symmetries and Conservation Laws

★ **Suppose physics is invariant under the transformation**

$$\psi \rightarrow \psi' = \hat{U}\psi$$ **e.g. rotation of the coordinate axes**

• **To conserve probability normalisation require**

$$\langle \psi | \psi \rangle = \langle \psi' | \psi' \rangle = \langle \hat{U}\psi | \hat{U}\psi \rangle = \langle \psi | \hat{U}^\dagger \hat{U} | \psi \rangle$$

$$\rightarrow \boxed{\hat{U}^\dagger \hat{U} = 1}$$ **i.e.** $\hat{U}$ **has to be unitary**

• **For physical predictions to be unchanged by the symmetry transformation, also require all QM matrix elements unchanged**

$$\boxed{\langle \psi | \hat{H} | \psi \rangle = \langle \psi' | \hat{H} | \psi' \rangle} = \langle \psi | \hat{U}^\dagger \hat{H} \hat{U} | \psi \rangle$$

i.e. require $$\hat{U}^\dagger \hat{H} \hat{U} = \hat{H}$$

$\times \hat{U}$ $$\hat{U}\hat{U}^\dagger \hat{H} \hat{U} = \hat{U}\hat{H} \rightarrow \hat{H}\hat{U} = \hat{U}\hat{H}$$

therefore $$\boxed{[\hat{H}, \hat{U}] = 0}$$ $\hat{U}$ **commutes with the Hamiltonian**

★ **Now consider the infinitesimal transformation** (ε **small**)

$$\hat{U} = 1 + i\varepsilon\hat{G}$$

($\hat{G}$ **is called the generator of the transformation)**

Symmetries and Conservation Laws

- For $\hat{U}$ to be unitary

$$\hat{U}\hat{U}^\dagger = (1+i\varepsilon\hat{G})(1-i\varepsilon\hat{G}^\dagger) = 1+i\varepsilon(\hat{G}-\hat{G}^\dagger)+O(\varepsilon^2)$$

neglecting terms in ε^2 $\quad UU^\dagger = 1 \quad \Rightarrow \quad \boxed{\hat{G} = \hat{G}^\dagger}$

i.e. $\hat{G}$ is Hermitian and therefore corresponds to an observable quantity G !

- Furthermore, $\quad [\hat{H},\hat{U}] = 0 \quad \Rightarrow [\hat{H}, 1+i\varepsilon\,\hat{G}] = 0 \quad \Rightarrow [\hat{H},\hat{G}] = 0$

But from QM $\quad \dfrac{\mathrm{d}}{\mathrm{d}t}\langle\hat{G}\rangle = i\langle[\hat{H},\hat{G}]\rangle = 0$

i.e. G is a conserved quantity.

Symmetry ⟷ Conservation Law

★ **For each symmetry of nature have an observable conserved quantity**

Example: Infinitesimal spatial translation $\quad x \rightarrow x+\varepsilon$

i.e. expect physics to be invariant under $\quad \psi(x) \rightarrow \psi' = \psi(x+\varepsilon)$

$$\psi'(x) = \psi(x+\varepsilon) = \psi(x) + \frac{\partial\psi}{\partial x}\varepsilon = \left(1+\varepsilon\frac{\partial}{\partial x}\right)\psi(x)$$

but $\quad \hat{p}_x = -i\frac{\partial}{\partial x} \quad \Rightarrow \quad \psi'(x) = (1+i\varepsilon\hat{p}_x)\psi(x)$

The generator of the symmetry transformation is $\hat{p}_x \quad \Rightarrow \quad p_x$ is conserved

- **Translational invariance of physics implies momentum conservation !**

Symmetries and Conservation Laws

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- In general the symmetry operation may depend on more than one parameter

$$\hat{U} = 1 + i\vec{\varepsilon}.\vec{G}$$

For example for an infinitesimal 3D linear translation : $\vec{r} \to \vec{r} + \vec{\varepsilon}$

$$\rightarrow \quad \hat{U} = 1 + i\vec{\varepsilon}.\vec{p} \qquad \vec{p} = (\hat{p}_x, \hat{p}_y, \hat{p}_z)$$

- So far have only considered an infinitesimal transformation, however a finite transformation can be expressed as a series of infinitesimal transformations

$$\hat{U}(\vec{\alpha}) = \lim_{n\to\infty}\left(1 + i\frac{\vec{\alpha}}{n}.\vec{G}\right)^n = e^{i\vec{\alpha}.\vec{G}}$$

<u>**Example:**</u> Finite spatial translation in 1D: $x \to x + x_0$ with $\hat{U}(x_0) = e^{ix_0\hat{p}_x}$

$$\begin{aligned}
\psi'(x) = \psi(x+x_0) &= \hat{U}\psi(x) = \exp\left(x_0\frac{\mathrm{d}}{\mathrm{d}x}\right)\psi(x) \qquad \left(p_x = -i\frac{\partial}{\partial x}\right) \\
&= \left(1 + x_0\frac{\mathrm{d}}{\mathrm{d}x} + \frac{x_0^2}{2!}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \ldots\right)\psi(x) \\
&= \psi(x) + x_0\frac{\mathrm{d}\psi}{\mathrm{d}x} + \frac{x_0^2}{2}\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + \ldots
\end{aligned}$$

i.e. obtain the expected Taylor expansion

Isospin

- The proton and neutron have very similar masses and the nuclear force is found to be approximately charge-independent, i.e.

$$V_{pp} \approx V_{np} \approx V_{nn}$$

- To reflect this symmetry, Heisenberg (1932) proposed that if you could "switch off" the electric charge of the proton

There would be no way to distinguish between a proton and neutron

- Proposed that the neutron and proton should be considered as two states of a single entity; the **nucleon**

$$p = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad n = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

★ Analogous to the spin-up/spin-down states of a spin-½ particle

ISOSPIN

★ Expect physics to be invariant under rotations in this space

- The neutron and proton form an isospin doublet with total isospin $I = \frac{1}{2}$ and third component $I_3 = \pm\frac{1}{2}$

Flavour Symmetry

We can extend this idea to the quarks:

★ **Assume the strong interaction treats all quark flavours equally (it does)**

- **Because $m_u \approx m_d$:**

  The strong interaction possesses an approximate flavour symmetry
  i.e. from the point of view of the strong interaction nothing changes if all up quarks are replaced by down quarks and *vice versa*.

- **Choose the basis**

$$u = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad d = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- **Express the invariance of the strong interaction under $u \leftrightarrow d$ as invariance under "rotations" in an abstract isospin space**

$$\begin{pmatrix} u' \\ d' \end{pmatrix} = \hat{U} \begin{pmatrix} u \\ d \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix}$$

The 2x2 unitary matrix depends on 4 complex numbers, i.e. 8 real parameters
But there are four constraints from $\hat{U}^{\dagger}\hat{U} = 1$

➡ **8 – 4 = 4 independent matrices**

- **In the language of group theory the four matrices form the U(2) group**

Flavour Symmetry

- One of the matrices corresponds to multiplying by a phase factor

$$\hat{U}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} e^{i\phi}$$

not a flavour transformation and of no relevance here.

- The remaining three matrices form an SU(2) group (special unitary) with $\boxed{\det U = 1}$
- For an infinitesimal transformation, in terms of the Hermitian generators $\hat{G}$

$$\hat{U} = 1 + i\varepsilon\hat{G}$$

- $\det U = 1 \quad \Rightarrow \quad Tr(\hat{G}) = 0$
- A linearly independent choice for $\hat{G}$ are the Pauli spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- The proposed flavour symmetry of the strong interaction has the same transformation properties as SPIN !
- Define ISOSPIN: $\vec{T} = \frac{1}{2}\vec{\sigma} \qquad \hat{U} = e^{i\vec{\alpha}.\vec{T}}$
- Check this works, for an infinitesimal transformation

$$\hat{U} = 1 + \frac{1}{2}i\vec{\varepsilon}.\vec{\sigma} \quad = \quad 1 + \frac{i}{2}(\varepsilon_1\sigma_1 + \varepsilon_2\sigma_2 + \varepsilon_3\sigma_3) = \begin{pmatrix} 1 + \frac{1}{2}i\varepsilon_3 & \frac{1}{2}i(\varepsilon_1 - i\varepsilon_2) \\ \frac{1}{2}i(\varepsilon_1 + i\varepsilon_2) & 1 - \frac{1}{2}i\varepsilon_3 \end{pmatrix}$$

Which is, as required, unitary and has unit determinant

$$U^\dagger U = I + O(\varepsilon^2) \qquad \det U = 1 + O(\varepsilon^2)$$

Properties of Isospin

- **Isospin has the exactly the same properties as spin**

$$[T_1, T_2] = iT_3 \quad [T_2, T_3] = iT_1 \quad [T_3, T_1] = iT_2$$

$$[T^2, T_3] = 0 \qquad T^2 = T_1^2 + T_2^2 + T_3^2$$

As in the case of spin, have three non-commuting operators, T_1, T_2, T_3 and even though all three correspond to observables, can't know them simultaneously. So label states in terms of total isospin I and the third component of isospin I_3

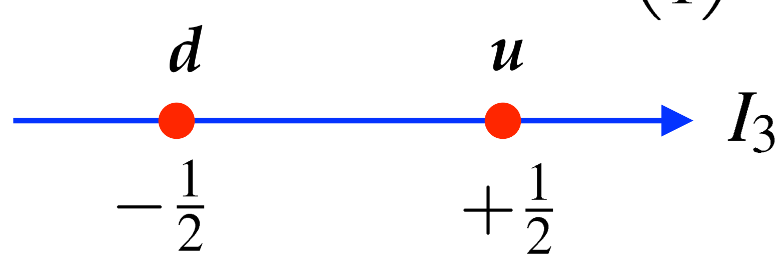
NOTE: isospin has nothing to do with spin – just the same mathematics

- **The eigenstates are exact analogues of the eigenstates of ordinary angular momentum** $|s, m\rangle \rightarrow |I, I_3\rangle$

  with $T^2|I, I_3\rangle = I(I+1)|I, I_3\rangle \qquad T_3|I, I_3\rangle = I_3|I, I_3\rangle$

- **In terms of isospin:**

$$u = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |\tfrac{1}{2}, +\tfrac{1}{2}\rangle \qquad d = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |\tfrac{1}{2}, -\tfrac{1}{2}\rangle$$



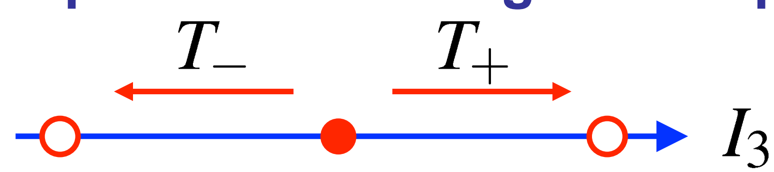
$$\boxed{I = \tfrac{1}{2}, \quad I_3 = \pm\tfrac{1}{2}}$$

- **In general** $I_3 = \frac{1}{2}(N_u - N_d)$

Properties of Isospin

- Can define isospin ladder operators – analogous to spin ladder operators

$\boxed{T_- \equiv T_1 - iT_2}$ u → d



$\boxed{T_+ \equiv T_1 + iT_2}$ d → u

$$T_+|I,I_3\rangle = \sqrt{I(I+1) - I_3(I_3+1)}|I,I_3+1\rangle$$
$$T_-|I,I_3\rangle = \sqrt{I(I+1) - I_3(I_3-1)}|I,I_3-1\rangle$$

Step up/down in I_3 until reach end of multiplet $T_+|I,+I\rangle = 0 \quad T_-|I,-I\rangle = 0$

$$\boxed{T_+u = 0 \qquad T_+d = u \qquad T_-u = d \qquad T_-d = 0}$$

- Ladder operators turn $u \to d$ and $d \to u$

★ Combination of isospin: e.g. what is the isospin of a system of two **d** quarks, is exactly analogous to combination of spin (i.e. angular momentum)

$$|I^{(1)}, I_3^{(1)}\rangle |I^{(2)}, I_3^{(2)}\rangle \to |I, I_3\rangle$$

- I_3 additive : $\quad I_3 = I_3^{(1)} + I_3^{(2)}$
- I in integer steps from $\quad |I^{(1)} - I^{(2)}|$ to $|I^{(1)} + I^{(2)}|$

★ Assumed symmetry of Strong Interaction under isospin transformations implies the existence of conserved quantites

- In strong interactions I_3 and I are conserved, analogous to conservation of J_z and J for angular momentum

SU(3) Flavour

★ Extend these ideas to include the strange quark. Since $m_s > m_u, m_d$ don't have an **exact symmetry**. But m_s not so very different from m_u, m_d and can treat the strong interaction (and resulting hadron states) as if it were symmetric under $u \leftrightarrow d \leftrightarrow s$

- **NOTE:** any results obtained from this assumption are only approximate as the symmetry is not exact.
- The assumed uds flavour symmetry can be expressed as

$$\begin{pmatrix} u' \\ d' \\ s' \end{pmatrix} = \hat{U} \begin{pmatrix} u \\ d \\ s \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{pmatrix} \begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

- The 3x3 unitary matrix depends on 9 complex numbers, i.e. 18 real parameters There are 9 constraints from $\hat{U}^\dagger \hat{U} = 1$

⇒ Can form 18 – 9 = 9 linearly independent matrices

These 9 matrices form a U(3) group.

- As before, one matrix is simply the identity multiplied by a complex phase and is of no interest in the context of flavour symmetry
- The remaining 8 matrices have $\det U = 1$ and form an SU(3) group
- The eight matrices (the Hermitian generators) are: $\vec{T} = \frac{1}{2}\vec{\lambda}$ $\quad \hat{U} = e^{i\vec{\alpha}.\vec{T}}$

SU(3) Flavour

★ In SU(3) flavour, the three quark states are represented by:

$$u = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad d = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad s = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

★ In SU(3) **uds** flavour symmetry contains SU(2) **ud** flavour symmetry which allows us to write the first three matrices:

$$\lambda_1 = \begin{pmatrix} \sigma_1 & & 0 \\ & & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} \sigma_2 & & 0 \\ & & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} \sigma_3 & & 0 \\ & & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

i.e. u ↔ d

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- The third component of isospin is now written $\boxed{I_3 = \frac{1}{2}\lambda_3}$

  with $I_3 u = +\frac{1}{2}u \quad I_3 d = -\frac{1}{2}d \quad I_3 s = 0$

- I_3 "counts the number of up quarks – number of down quarks in a state
- As before, ladder operators $T_\pm = \frac{1}{2}(\lambda_1 \pm i\lambda_2)$ $\quad d \xleftrightarrow{T_\pm} u$

SU(3) Flavour

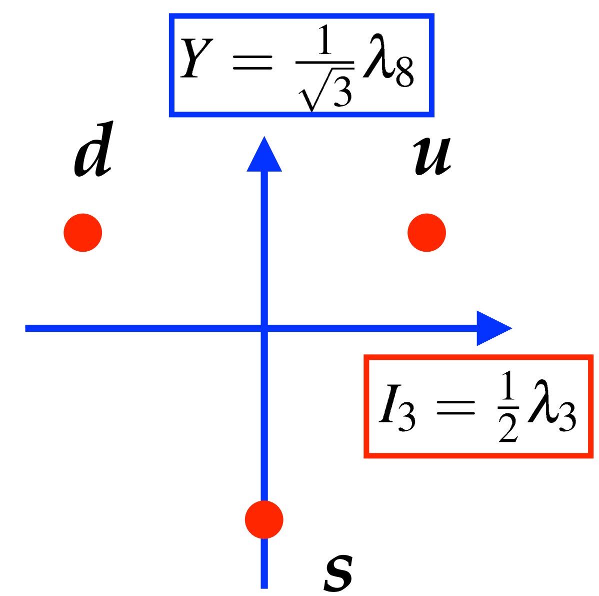
- **Now consider the matrices corresponding to the u ↔ s and d ↔ s**

u ↔ s	$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$
d ↔ s	$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

- **Hence in addition to** $\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ **have two other traceless diagonal matrices**
- **However the three diagonal matrices are not be independent.**
- **Define the eighth matrix, λ_8, as the linear combination:**

$$\lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} + \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

which specifies the "vertical position" in the 2D plane

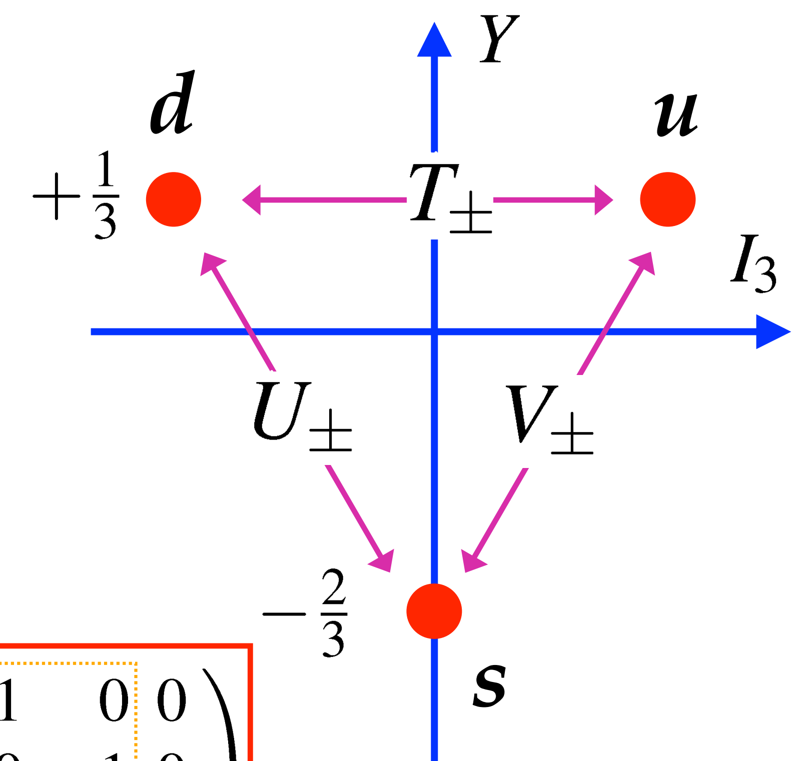


"Only need two axes (quantum numbers) to specify a state in the 2D plane": (I_3,Y)

SU(3) Flavour

★ **The other six matrices form six ladder operators which step between the states**

$$T_\pm = \tfrac{1}{2}(\lambda_1 \pm i\lambda_2)$$
$$V_\pm = \tfrac{1}{2}(\lambda_4 \pm i\lambda_5)$$
$$U_\pm = \tfrac{1}{2}(\lambda_6 \pm i\lambda_7)$$



with $I_3 = \tfrac{1}{2}\lambda_3$ $\quad Y = \tfrac{1}{\sqrt{3}}\lambda_8$

and the eight Gell-Mann matrices

u ↔ d

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

u ↔ s

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$

d ↔ s

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$\lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

QCD

★ Represent r, g, b SU(3) colour states by:

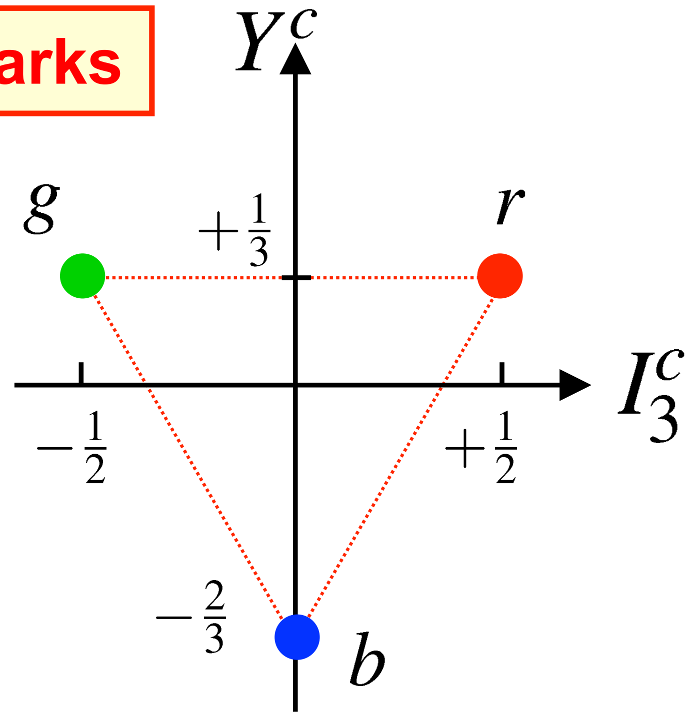
$$r = \begin{pmatrix}1\\0\\0\end{pmatrix}; \quad g = \begin{pmatrix}0\\1\\0\end{pmatrix}; \quad b = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

★ Colour states can be labelled by two quantum numbers:
- I_3^c colour isospin
- Y^c colour hypercharge

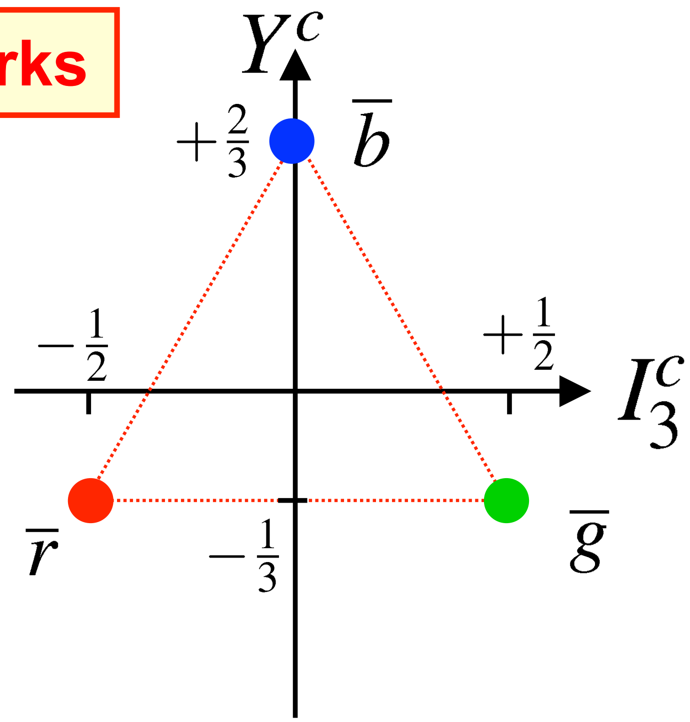
Exactly analogous to labelling u,d,s flavour states by I_3 and Y

★ Each quark (anti-quark) can have the following colour quantum numbers:

quarks



anti-quarks



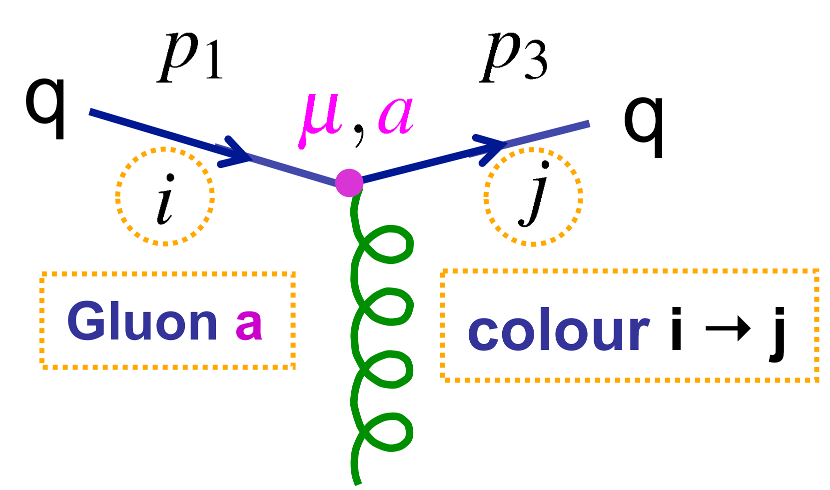
Quark-gluon interaction

- Representing the colour part of the fermion wave-functions by:

$$r = c_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad g = c_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad b = c_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

- Particle wave-functions $u(p) \longrightarrow c_i u(p)$
- The QCD qqg vertex is written:

$$\overline{u}(p_3)c_j^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_i u(p_1)$$



- Only difference w.r.t. QED is the insertion of the 3x3 SU(3) Gell-Mann matrices
- Isolating the colour part:

$$c_j^\dagger \lambda^a c_i = c_j^\dagger \begin{pmatrix} \lambda_{1i}^a \\ \lambda_{2i}^a \\ \lambda_{3i}^a \end{pmatrix} = \lambda_{ji}^a$$

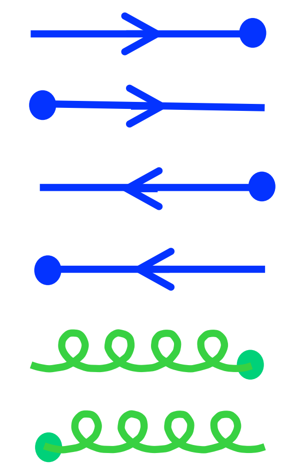
- Hence the fundamental quark - gluon QCD interaction can be written

$$\overline{u}(p_3)c_j^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_i u(p_1) \equiv \overline{u}(p_3)\{-\tfrac{1}{2}ig_s\lambda_{ji}^a\gamma^\mu\}u(p_1)$$

Feynman Rules for QCD

- **External Lines**

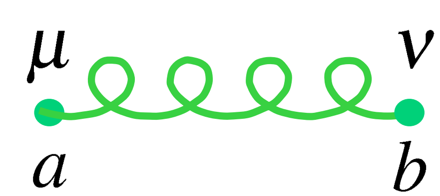
spin 1/2	incoming quark	$u(p)$	
	outgoing quark	$\overline{u}(p)$	
	incoming anti-quark	$\overline{v}(p)$	
	outgoing anti-quark	$v(p)$	
spin 1	incoming gluon	$\varepsilon^{\mu}(p)$	
	outgoing gluon	$\varepsilon^{\mu}(p)^{*}$	



- **Internal Lines (propagators)**

spin 1 gluon

$$\boxed{\frac{-ig_{\mu\nu}}{q^2}\delta^{ab}}$$

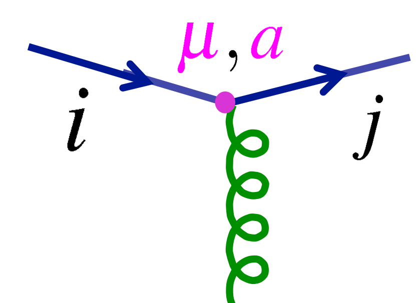


a, b = 1,2,…,8 are gluon colour indices

- **Vertex Factors**

spin 1/2 quark

$$\boxed{-ig_s\tfrac{1}{2}\lambda^{a}_{ji}\gamma^{\mu}}$$



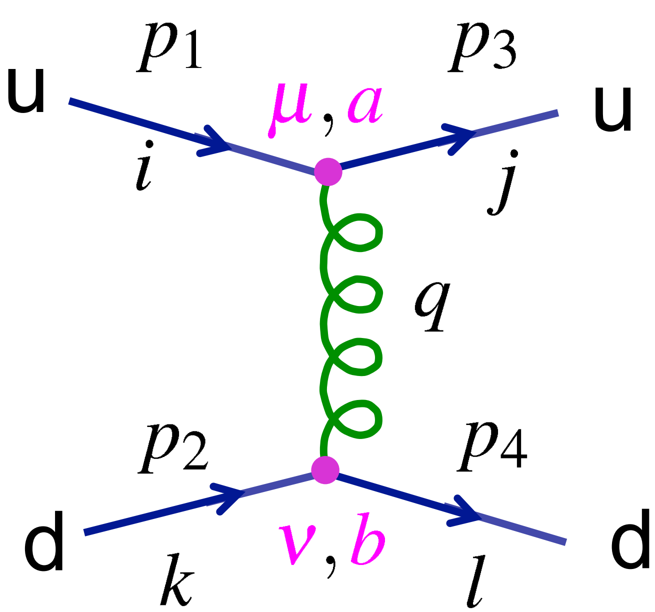
i, j = 1,2,3 are quark colours,

λ^a a = 1,2,..8 are the Gell-Mann SU(3) matrices

- **+ 3 gluon and 4 gluon interaction vertices**
- **Matrix Element** $-iM$ = **product of all factors**

QCD Scattering

★ **Consider QCD scattering of an up and a down quark**



- **The incoming and out-going quark colours are labelled by** $i, j, k, l = \{1,2,3\}$ (or $\{r, g, b\}$)
- **In terms of colour this scattering is**
$$ik \rightarrow jl$$
- **The 8 different gluons are accounted for by the colour indices** $a, b = 1, 2, \ldots, 8$
- **NOTE: the δ-function in the propagator ensures** $a = b$**, i.e. the gluon "emitted" at *a* is the same as that "absorbed" at *b***

★ **Applying the Feynman rules:**

$$-iM = \left[\overline{u}_u(p_3)\{-\tfrac{1}{2}ig_s\lambda^a_{ji}\gamma^\mu\}u_u(p_1)\right]\frac{-ig_{\mu\nu}}{q^2}\delta^{ab}\left[\overline{u}_d(p_4)\{-\tfrac{1}{2}ig_s\lambda^b_{lk}\gamma^\nu\}u_d(p_2)\right]$$

where summation over *a* and *b* (and μ and ν) is implied.

★ **Summing over *a* and *b* using the δ-function gives:**

$$\boxed{M = -\frac{g_s^2}{4}\underbrace{\lambda^a_{ji}\lambda^a_{lk}}_{\text{Sum over all 8 gluons (repeated indices)}}\frac{1}{q^2}g_{\mu\nu}[\overline{u}_u(p_3)\gamma^\mu u_u(p_1)][\overline{u}_d(p_4)\gamma^\nu u_d(p_2)]}$$

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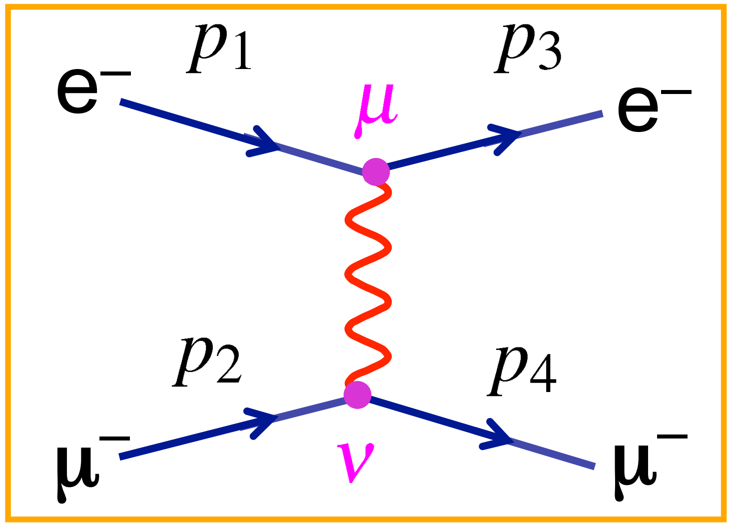
QED vs QCD

QED

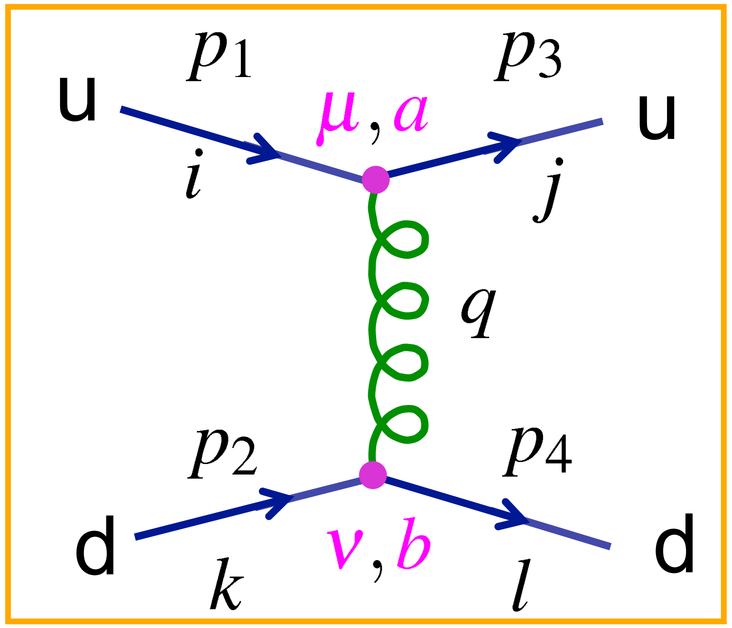
$$-iM = [\overline{u}(p_3)ie\gamma^{\mu}u(p_1)]\frac{-ig_{\mu\nu}}{q^2}[\overline{u}(p_4)ie\gamma^{\nu}u(p_2)]$$

$$\boxed{M = -e^2\frac{1}{q^2}g_{\mu\nu}[\overline{u}(p_3)\gamma^{\mu}u(p_1)][\overline{u}(p_4)\gamma^{\nu}u(p_2)]}$$



QCD

$$\boxed{M = -\frac{g_s^2}{4}\lambda^a_{ji}\lambda^a_{lk}\frac{1}{q^2}g_{\mu\nu}[\overline{u}_u(p_3)\gamma^{\mu}u_u(p_1)][\overline{u}_d(p_4)\gamma^{\nu}u_d(p_2)]}$$



★ **QCD Matrix Element = QED Matrix Element with:**

- $\boxed{e^2 \rightarrow g_s^2}$ **or equivalently** $\boxed{\alpha = \frac{e^2}{4\pi} \rightarrow \alpha_s = \frac{g_s^2}{4\pi}}$

+ QCD Matrix Element includes an additional "colour factor"

$$\boxed{C(ik \rightarrow jl) \equiv \frac{1}{4}\sum_{a=1}^{8}\lambda^a_{ji}\lambda^a_{lk}}$$

QCD Color Factor

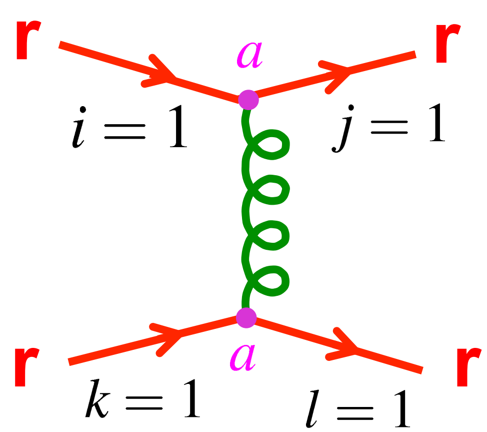
- **QCD colour factors reflect the gluon states that are involved**

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda^8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Gluons: $r\overline{g}\,, g\overline{r}$ | $r\overline{b}\,, b\overline{r}$ | $g\overline{b}\,, b\overline{g}$ | $\frac{1}{\sqrt{2}}(r\overline{r}-g\overline{g})$ $\frac{1}{\sqrt{6}}(r\overline{r}+g\overline{g}-2b\overline{b})$

➊ Configurations involving a single colour



QCD Color Factor

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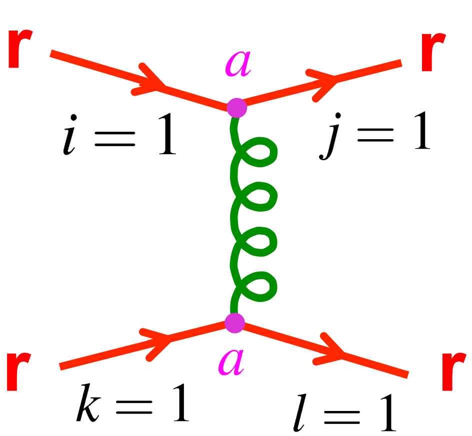
- QCD colour factors reflect the gluon states that are involved

$$\lambda^1=\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}\quad \lambda^4=\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}\quad \lambda^6=\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}\quad \lambda^3=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix}$$

$$\lambda^2=\begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix}\quad \lambda^5=\begin{pmatrix}0&0&-i\\0&0&0\\i&0&0\end{pmatrix}\quad \lambda^7=\begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix}\quad \lambda^8=\frac{1}{\sqrt{3}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix}$$

Gluons: $r\bar{g},\, g\bar{r}$ | $r\bar{b},\, b\bar{r}$ | $g\bar{b},\, b\bar{g}$ | $\frac{1}{\sqrt{2}}(r\bar{r}-g\bar{g})$ $\frac{1}{\sqrt{6}}(r\bar{r}+g\bar{g}-2b\bar{b})$

➊ Configurations involving a single colour



- Only matrices with non-zero entries in **11** position are involved

$$C(rr\to rr) = \frac{1}{4}\sum_{a=1}^{8}\lambda^a_{11}\lambda^a_{11} = \frac{1}{4}(\lambda^3_{11}\lambda^3_{11}+\lambda^8_{11}\lambda^8_{11})$$

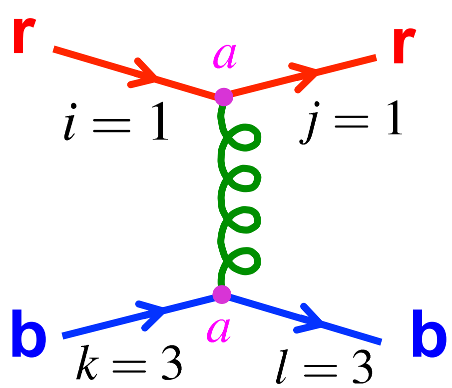
$$= \frac{1}{4}\left(1+\frac{1}{3}\right)=\frac{1}{3}$$

Similarly find

$$\boxed{C(rr\to rr)=C(gg\to gg)=C(bb\to bb)=\frac{1}{3}}$$

QCD Color Factor

❷ Other configurations where quarks don't change colour   e.g. $rb \rightarrow rb$

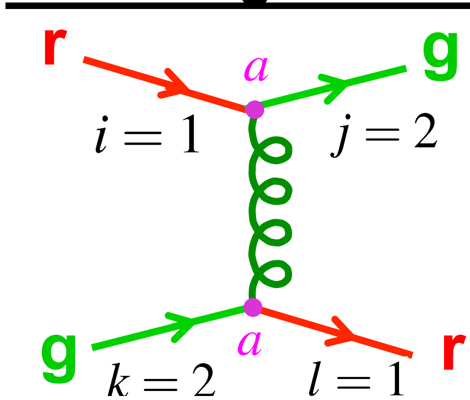


- Only matrices with non-zero entries in **11** and **33** position are involved

$$C(rb \rightarrow rb) = \frac{1}{4}\sum_{a=1}^{8} \lambda_{11}^{a}\lambda_{33}^{a} = \frac{1}{4}(\lambda_{11}^{8}\lambda_{33}^{8})$$
$$= \frac{1}{4}\left(\frac{1}{\sqrt{3}}.\frac{-2}{\sqrt{3}}\right) = -\frac{1}{6}$$

Similarly $C(rb \rightarrow rb) = C(rg \rightarrow rg) = C(gr \rightarrow gr) = C(gb \rightarrow gb) = C(br \rightarrow br) = C(bg \rightarrow bg) = -\frac{1}{6}$

❸ Configurations where quarks swap colours   e.g. $rg \rightarrow gr$



- Only matrices with non-zero entries in **12** and **21** position are involved

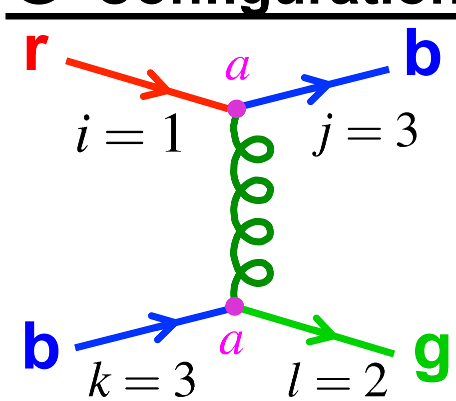
Gluons $r\bar{g}, g\bar{r}$

$$C(rg \rightarrow gr) = \frac{1}{4}\sum_{a=1}^{8} \lambda_{21}^{a}\lambda_{12}^{a} = \frac{1}{4}(\lambda_{21}^{1}\lambda_{12}^{1} + \lambda_{21}^{2}\lambda_{12}^{2})$$
$$= \frac{1}{4}(i(-i)+1) = \frac{1}{2}$$

$\hat{T}_{+}^{(ij)}\hat{T}_{-}^{(kl)}$

$$C(rb \rightarrow br) = C(rg \rightarrow gr) = C(gr \rightarrow rg) = C(gb \rightarrow bg) = C(br \rightarrow rb) = C(bg \rightarrow gb) = \frac{1}{2}$$

❹ Configurations involving 3 colours   e.g. $rb \rightarrow bg$



- Only matrices with non-zero entries in the **13** and **32** position
- But none of the λ matrices have non-zero entries in the **13** and **32** positions. Hence the colour factor is zero

★ colour is conserved

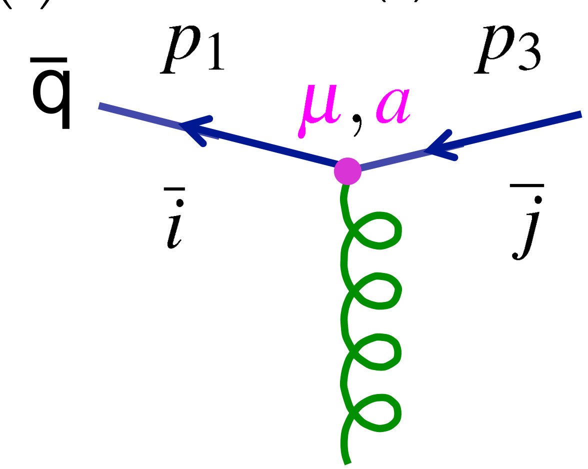
QCD Color Factor (anti-quark)

- **Recall the colour part of wave-function:**

$$r = c_1 = \begin{pmatrix}1\\0\\0\end{pmatrix} \qquad g = c_2 = \begin{pmatrix}0\\1\\0\end{pmatrix} \qquad b = c_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

- **The QCD** qqg **vertex was written:**

$$\overline{u}(p_3)c_j^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_i u(p_1)$$



★ **Now consider the anti-quark vertex**

- **The QCD** q̄q̄g **vertex is:**

$$\overline{v}(p_1)c_i^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_j v(p_3)$$

Note that the incoming anti-particle now enters on the LHS of the expression

- **For which the colour part is**

$$c_i^\dagger\lambda^a c_j = c_i^\dagger\begin{pmatrix}\lambda^a_{1j}\\ \lambda^a_{2j}\\ \lambda^a_{3j}\end{pmatrix} = \lambda^a_{ij}$$

i.e indices *ij* are swapped with respect to the quark case

- **Hence**

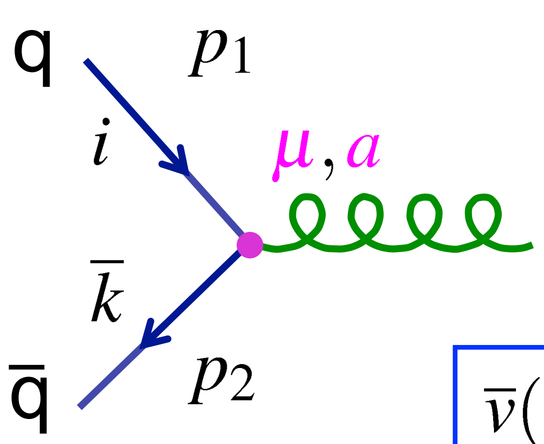
$$\boxed{\overline{v}(p_1)c_i^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_j v(p_3) \equiv \overline{v}(p_1)\{-\tfrac{1}{2}ig_s\lambda^a_{ij}\gamma^\mu\}v(p_3)}$$

- **c.f. the quark - gluon QCD interaction**

$$\overline{u}(p_3)c_j^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_i u(p_1) \equiv \overline{u}(p_3)\{-\tfrac{1}{2}ig_s\lambda^a_{ji}\gamma^\mu\}u(p_1)$$

Quark-Antiquark annihilation

★ **Finally we can consider the quark – anti-quark annihilation**



QCD vertex: $\overline{v}(p_2)c_k^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_i u(p_1)$

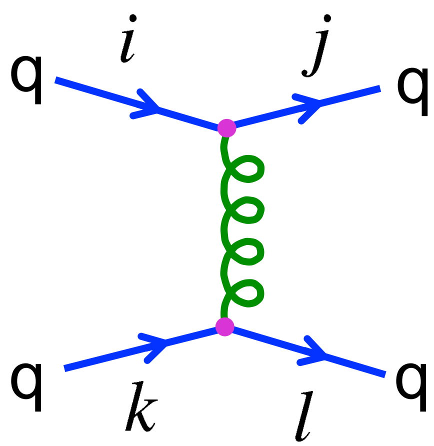
with $c_k^\dagger\lambda^a c_i = \lambda^a_{ki}$

$$\boxed{\overline{v}(p_2)c_k^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_i u(p_1) \equiv \overline{v}(p_2)\{-\tfrac{1}{2}ig_s\lambda^a_{ki}\gamma^\mu\}u(p_1)}$$

QCD Color Factor

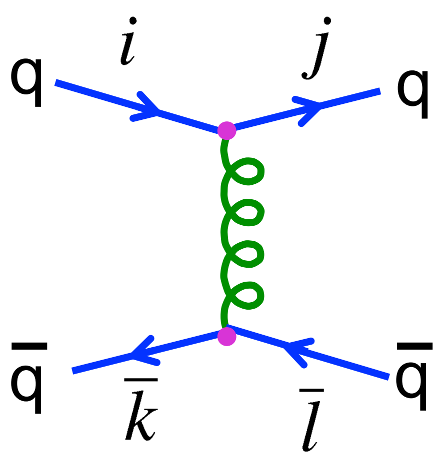
- **Consequently the colour factors for the different diagrams are:**

e.g.



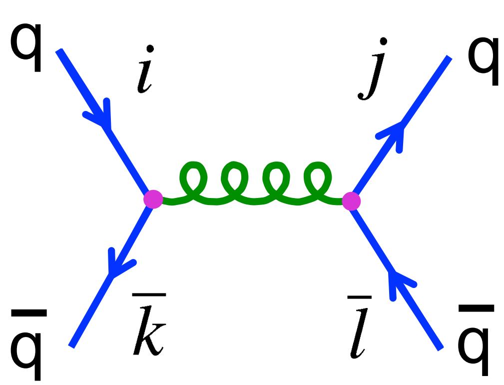
$$C(ik \to jl) \equiv \frac{1}{4}\sum_{a=1}^{8} \lambda^a_{ji}\lambda^a_{lk}$$

$$C(rr \to rr) = \tfrac{1}{3}$$
$$C(rg \to rg) = -\tfrac{1}{6}$$
$$C(rg \to gr) = \tfrac{1}{2}$$



$$C(i\bar{k} \to j\bar{l}) \equiv \frac{1}{4}\sum_{a=1}^{8} \lambda^a_{ji}\lambda^a_{kl}$$

$$C(r\bar{r} \to r\bar{r}) = \tfrac{1}{3}$$
$$C(r\bar{g} \to r\bar{g}) = -\tfrac{1}{6}$$
$$C(r\bar{r} \to g\bar{g}) = \tfrac{1}{2}$$



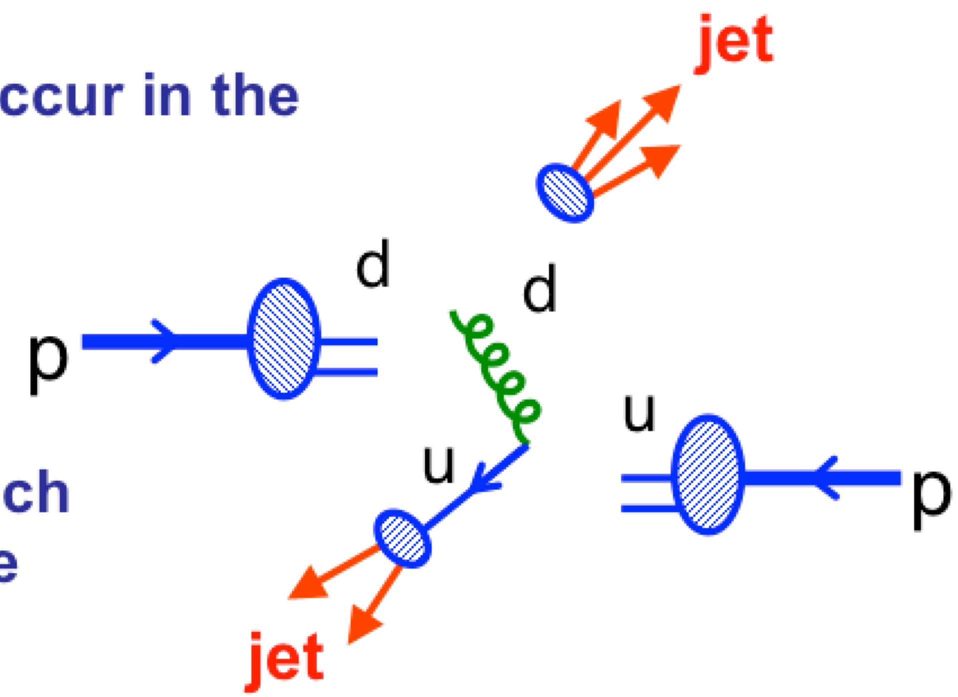
$$C(i\bar{k} \to j\bar{l}) \equiv \frac{1}{4}\sum_{a=1}^{8} \lambda^a_{ki}\lambda^a_{jl}$$

$$C(r\bar{r} \to r\bar{r}) = \tfrac{1}{3}$$
$$C(r\bar{g} \to r\bar{g}) = \tfrac{1}{2}$$
$$C(r\bar{r} \to g\bar{g}) = -\tfrac{1}{6}$$

Colour index of adjoint spinor comes first

Quark-antiquark scattering

- Consider the process $u+d \rightarrow u+d$ which can occur in the high energy proton-proton scattering



- There are nine possible colour configurations of the colliding quarks which are all equally likely.
- Need to determine the average matrix element which is the sum over all possible colours divided by the number of possible initial colour states

$$\langle |M_{fi}|^2 \rangle = \frac{1}{3} \cdot \frac{1}{3} \sum_{i,j,k,l=1}^{3} |M_{fi}(ij \rightarrow kl)|^2$$

- The colour average matrix element contains the average colour factor

$$\langle |C|^2 \rangle = \frac{1}{9} \sum_{i,j,k,l=1}^{3} |C(ij \rightarrow kl)|^2$$

- For $qq \rightarrow qq$

$rr \rightarrow rr,..$ $\quad$ $rb \rightarrow rb,..$ $\quad$ $rb \rightarrow br,..$

$$\langle |C|^2 \rangle = \frac{1}{9}\left[3 \times \left(\frac{1}{3}\right)^2 + 6 \times \left(-\frac{1}{6}\right)^2 + 6 \times \left(\frac{1}{2}\right)^2\right] = \frac{2}{9}$$

Quark-antiquark scattering

- Previously derived the Lorentz Invariant cross section for $e^-\mu^- \to e^-\mu^-$ elastic scattering in the ultra-relativistic limit (handout 6).

QED

$$\frac{d\sigma}{dq^2} = \frac{2\pi\alpha^2}{q^4}\left[1+\left(1+\frac{q^2}{s}\right)^2\right]$$

- For $ud \to ud$ in QCD replace $\alpha \to \alpha_s$ and multiply by $\langle |C|^2 \rangle$

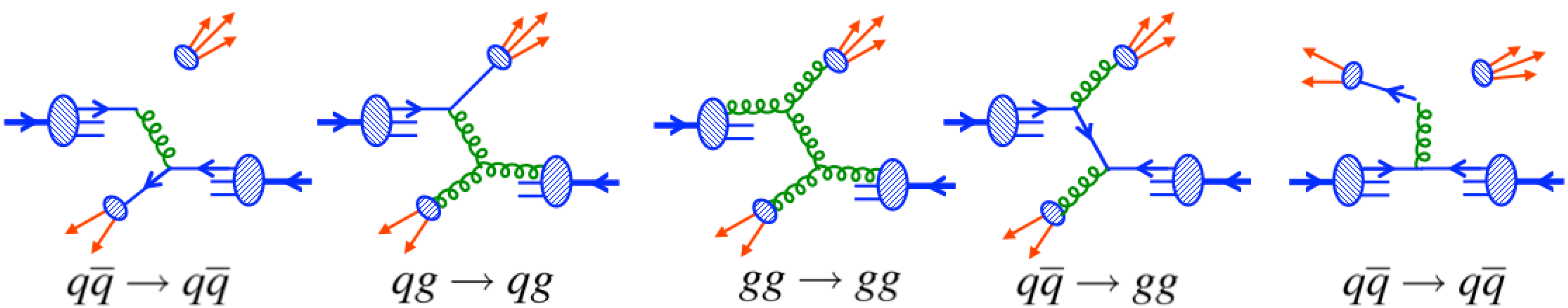
QCD

$$\frac{d\sigma}{dq^2} = \frac{2}{9}\frac{2\pi\alpha_S^2}{q^4}\left[1+\left(1+\frac{q^2}{\hat{s}}\right)^2\right]$$

Never see colour, but enters through colour factors. Can tell QCD is SU(3)

- Here $\hat{s}$ is the centre-of-mass energy of the quark-quark collision
- The calculation of hadron-hadron scattering is very involved, need to include parton structure functions and include all possible interactions

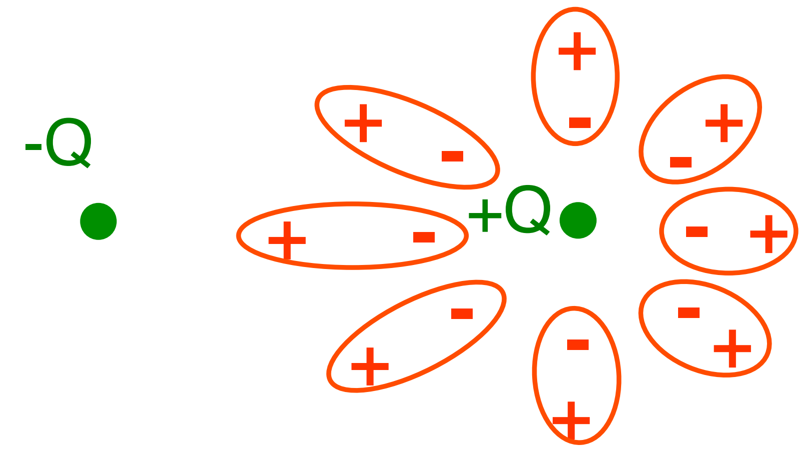
e.g. two jet production in proton-antiproton collisions



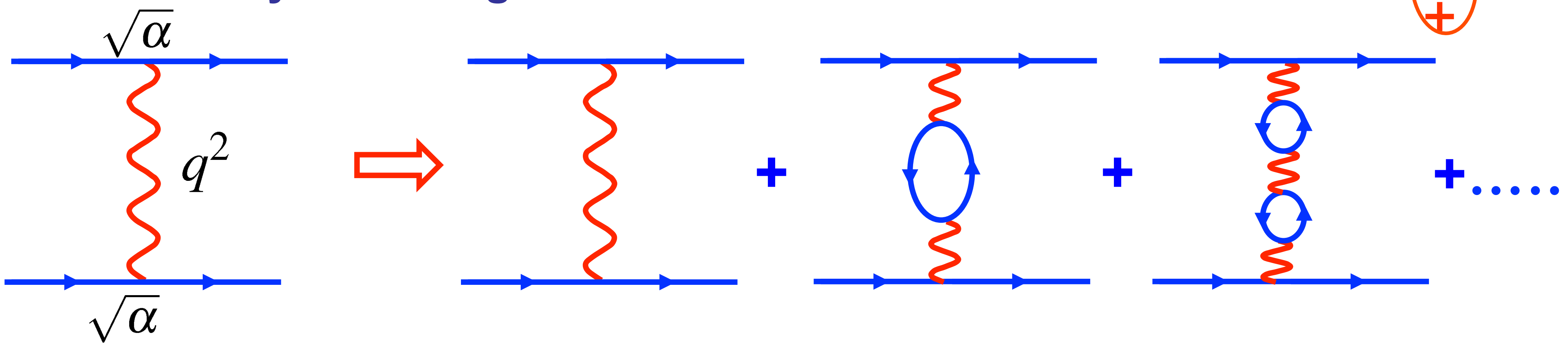
Running Coupling Constant

QED

- "bare" charge of electron screened by virtual e^+e^- pairs
- behaves like a polarizable dielectric

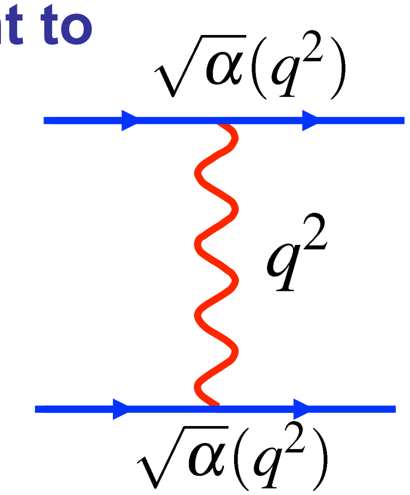


★ In terms of Feynman diagrams:



★ Same final state so add matrix element **amplitudes:** $M = M_1 + M_2 + M_3 + \ldots$

★ Giving an infinite series which can be summed and is equivalent to a single diagram with "running" coupling constant

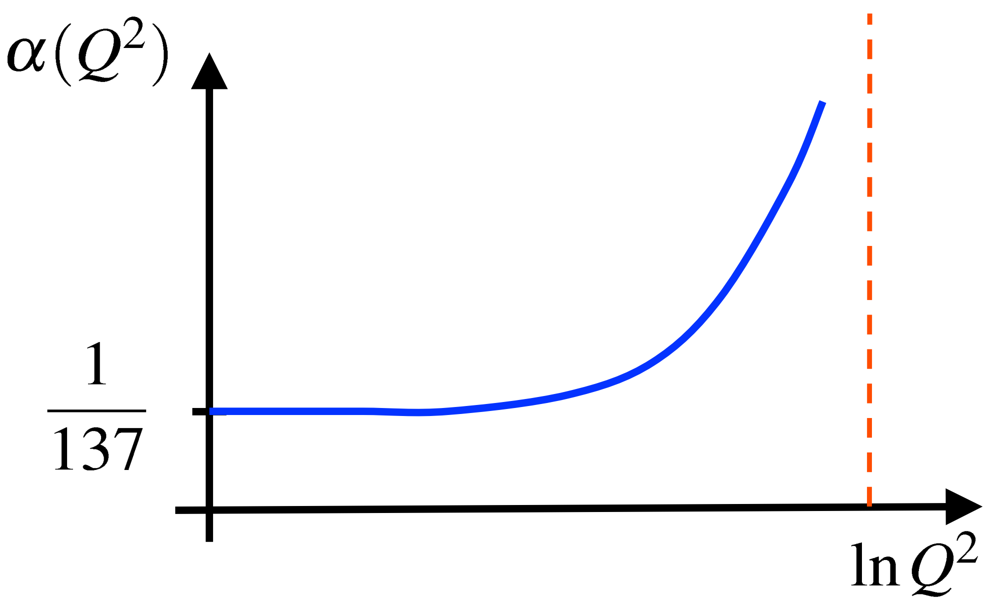


$$\alpha(Q^2) = \alpha(Q_0^2) \Bigg/ \left[1 - \frac{\alpha(Q_0^2)}{3\pi} \ln\left(\frac{Q^2}{Q_0^2}\right)\right]$$

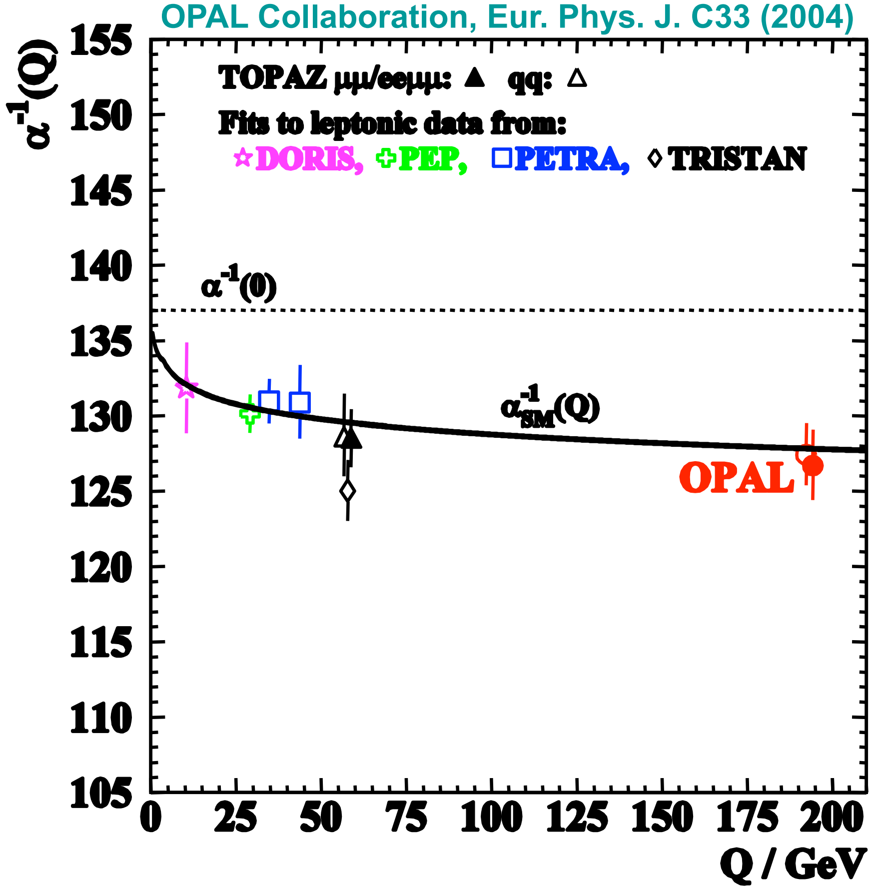
$$Q^2 \gg Q_0^2$$

Note sign

Running Coupling Constant



OPAL Collaboration, Eur. Phys. J. C33 (2004)



★ **Might worry that coupling becomes infinite at**

$$\ln\left(\frac{Q^2}{Q_0^2}\right) = \frac{3\pi}{1/137}$$

i.e. at $Q \sim 10^{26}\,\mathrm{GeV}$

- **But quantum gravity effects would come in way below this energy and it is highly unlikely that QED "as is" would be valid in this regime**

★ **In QED, running coupling increases very slowly**

- **Atomic physics:** $Q^2 \sim 0$

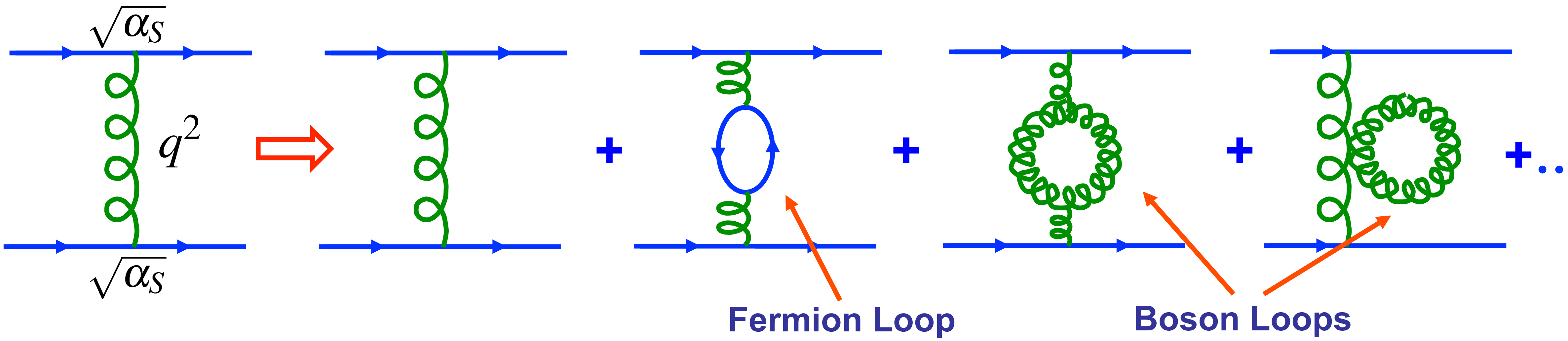
$$1/\alpha = 137.03599976(50)$$

- **High energy physics:**

$$1/\alpha(193\,\mathrm{GeV}) = 127.4 \pm 2.1$$

Running Coupling Constant

QCD Similar to QED but also have gluon loops



- ★ Remembering **adding amplitudes**, so can get negative interference and the sum can be smaller than the original diagram alone
- ★ Bosonic loops "interfere negatively"

$$\alpha_S(Q^2) = \alpha_S(Q_0^2) \Bigg/ \left[1 + B\alpha_S(Q_0^2)\ln\left(\frac{Q^2}{Q_0^2}\right)\right]$$

with $$B = \frac{11N_c - 2N_f}{12\pi}$$

N_c = no. of colours
N_f = no. of quark flavours

$N_c = 3;\ N_f = 6$ ➡ $B > 0$

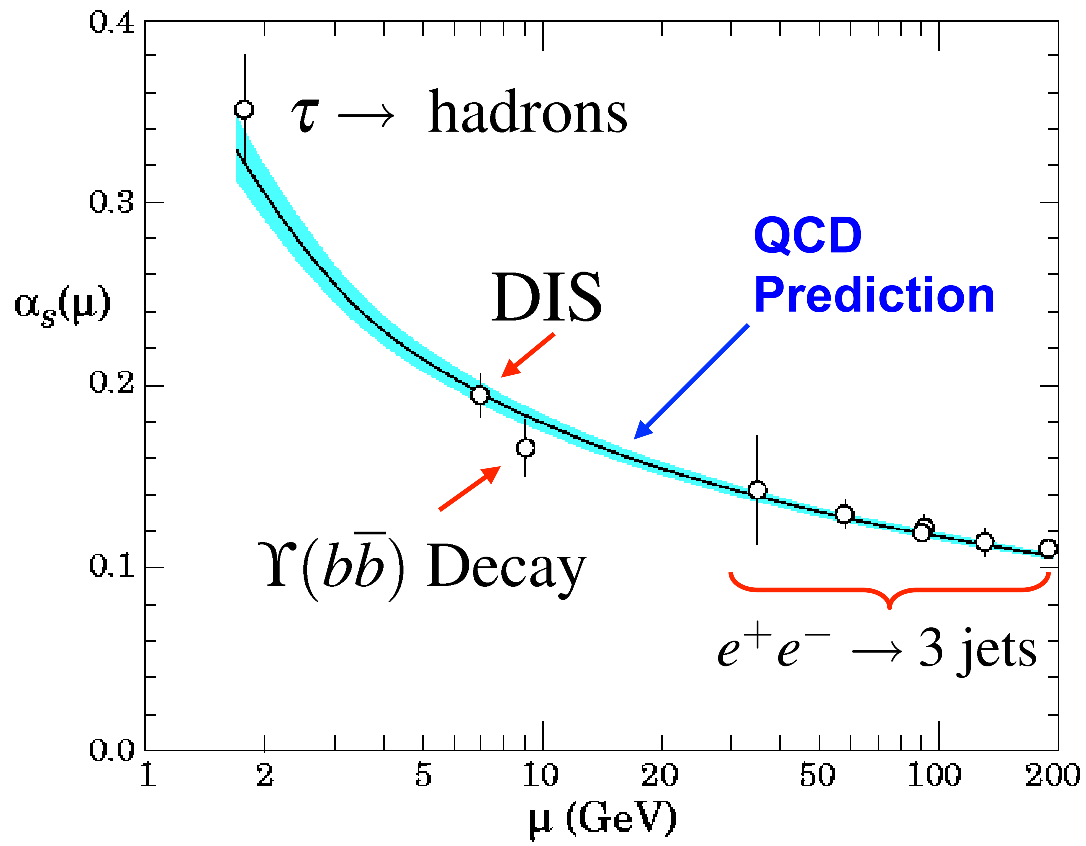
➡ $\boxed{\alpha_S \text{ decreases with } Q^2}$

Nobel Prize for Physics, 2004
(Gross, Politzer, Wilczek)

Running Coupling Constant

- ★ Measure α_s in many ways:
  - jet rates
  - DIS
  - tau decays
  - bottomonium decays
  - +…

★ As predicted by QCD, α_s decreases with Q^2



$\mu \approx Q$

★ At low Q^2: α_s is large, e.g. at $Q^2 = 1\,\mathrm{GeV}^2$ find $\alpha_s \sim 1$

- Can't use perturbation theory ! This is the reason why QCD calculations at low energies are so difficult, e.g. properties hadrons, hadronisation of quarks to jets,…

★ At high Q^2 : α_s is rather small, e.g. at $Q^2 = M_Z^2$ find $\alpha_s \sim 0.12$

➡ **Asymptotic Freedom**

- Can use perturbation theory and this is the reason that in DIS at high Q^2 quarks behave as if they are quasi-free (i.e. only weakly bound within hadrons)

Running Coupling Constant

★ Superficially QCD very similar to QED
★ But gluon self-interactions are believed to result in colour confinement
★ All hadrons are colour singlets which explains why only observe

Mesons **Baryons**

★ A low energies $\alpha_S \sim 1$

➡ Can't use perturbation theory !

Non-Perturbative regime

★ Coupling constant runs, smaller coupling at higher energy scales

$$\alpha_S(100\,\mathrm{GeV}) \sim 0.1$$

➡ Can use perturbation theory

Asymptotic Freedom

★ **Where calculations can be performed, QCD provides a good description of relevant experimental data**