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General Relativity
Sheet 5

HS 2019
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Exercise 1 [Calculations on a Manifold]

Consider $\mathbb{R}^3$ as a manifold with the flat Euclidean metric, and coordinates $\{x, y, z\}$. Introduce spherical coordinates $\{r, \theta, \phi\}$ related to $\{x, y, z\}$ by

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}$$

so that the metric takes the form $\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2$.

a) A particle moves along a parameterized curve given by

$$\begin{aligned} x(\lambda) &= \cos\lambda \\ y(\lambda) &= \sin\lambda \\ z(\lambda) &= \lambda \end{aligned}$$

Express the path of this curve in the $\{r, \theta, \phi\}$ system.

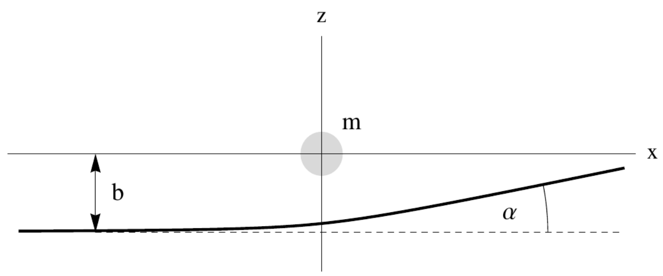
b) Calculate the components of the tangent vector to the curve in both the Cartesian and spherical coordinate system.

Exercise 2 [Christoffel symbols for diagonal metric tensor]

For a diagonal metric, show that the Christoffel symbols are given by

$$\begin{aligned} \Gamma^{\lambda}_{\mu\nu} &= 0, & \Gamma^{\lambda}_{\mu\mu} &= \frac{-1}{2g_{\lambda\lambda}}\partial_\lambda g_{\mu\mu}, \\ \Gamma^{\lambda}_{\lambda\mu} &= \partial_\mu \log\sqrt{|g_{\lambda\lambda}|}, & \Gamma^{\mu}_{\mu\mu} &= \partial_\mu \log\sqrt{|g_{\mu\mu}|}, \end{aligned} \qquad (1)$$

where $\mu \neq \nu \neq \lambda$ and there is no summation over repeated indices.



Exercise 3 [Newtonian limit of gravity and deflection of light]

In the weak field limit of general relativity, the metric of the static gravitational field can be written

$$ds^2 = \left(1 + \frac{2\phi}{c^2}\right) c^2 dt^2 - \left(1 - \frac{2\phi}{c^2}\right) \left(dx^2 + dy^2 + dz^2\right), \tag{2}$$

where ϕ depends on position, but not on time.

a) Use eqns. (1) to find the Christoffel symbols of this metric to linear order in $\frac{\phi}{c^2}$ (and $\frac{\partial_i \phi}{c^2}$).

For a concrete application, we consider the motion of slow test particles as well as photons in the weak gravitational field of a static star. In the following, $\phi(r) = -\frac{Gm}{r}$, where G is Newton's gravitational constant, m is the mass of the star, and r is the distance from the star.

b) Consider the equation of motion for a massive test-particle with velocity $\mathbf{v}$ that satisfies $|\mathbf{v}| \ll c$. Show that to lowest order in $\mathbf{v}^2/c^2$ the geodesic equation reduces to the non-relativistic equation of motion in the Newtonian potential $\phi(r)$!

c) Now we examine the trajectory of a photon in the same field $\phi(r)$. Use now units where $c = 1$. Assume that its initial four-momentum is $\kappa^\mu = (p, p, 0, 0)$, and that the change in momentum l^μ is small, so that the equation of motion can be expanded to leading order in ϕ and l^μ (note that both $\partial_i \phi$ and l^μ are $O\left(\phi^1\right)$, whereas $\kappa^\mu = O\left(\phi^0\right)$). Thus determine the deflection angle

$$\alpha = \frac{l^z}{p} + O\left(\left(l^z/p\right)^2\right) \tag{3}$$

as a function of the impact parameter b (see figure) to leading order in ϕ.

Hint: Choose the affine parameter λ such that $\frac{dx^\mu}{d\lambda} = p^\mu$. In the lowest order computation it is sufficient to integrate the equation of motion for l^μ along the unperturbed worldline determined by the zeroth order equation for $\frac{d}{d\lambda} p^\mu$.