Elementary Particle Theory – PHY452

Fall Semester 2020

Exercise Sheet 1

Exercise 1: Scalar field theory

A string of length a, mass per unit length σ and tension T is fixed at each end. The Lagrangian governing the time evolution of the transverse displacement $\phi(x,t)$ is

$$L = \int_0^a dx \mathcal{L} = \int_0^a dx \left[\frac{\sigma}{2} \left(\frac{\partial \phi}{\partial t} \right)^2 - \frac{T}{2} \left(\frac{\partial \phi}{\partial x} \right)^2 \right] , \tag{1}$$

where x identifies position along the string from one end point. From the principle of least action one can derive the non relativistic Euler-Lagrange field equations

$$\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial x} \left(\frac{\partial \mathcal{L}}{\partial(\partial \phi / \partial x)} \right) - \frac{\partial}{\partial t} \left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \right) = 0 . \tag{2}$$

a) Apply Eq. (2) to derive the equation of motions for the field ϕ. What kind of motion does the string have?

b) By expressing the displacement as a sine series Fourier expansion in the form

$$\phi(x,t) = \sqrt{\frac{2}{a}} \sum_{n=1}^{\infty} \sin \left(\frac{n\pi x}{a} \right) q_n(t) \tag{3}$$

show that the Lagrangian becomes

$$L = \sum_{n=1}^{\infty} \left[\frac{\sigma}{2} \dot{q}_n^2 - \frac{T}{2} \left(\frac{n\pi}{a} \right)^2 q_n^2 \right] . \tag{4}$$

Derive the equations of motion and show that the string is equivalent to an infinite set of decoupled harmonic oscillators with frequencies

$$\omega_n = \sqrt{\frac{T}{\sigma}} \left(\frac{n\pi}{a} \right) . \tag{5}$$

Exercise 2: Noether's theorem

The motion of a complex field $\phi(x)$ is governed by the Lorentz invariant Lagrangian density

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi - \frac{\lambda}{2} (\phi^* \phi)^2 . \tag{6}$$

Write down the Euler-Lagrange equations of motion for this system. Verify that the Lagrangian is invariant under the infinitesimal transformation

$$\delta \phi = i \alpha \phi , \qquad \delta \phi^* = -i \alpha \phi^* . \tag{7}$$

Derive the Noether current associated with this transformation and verify explicitly that it is conserved using the field equations satisfied by ϕ.

Exercise 3: Two-body decay

Consider the decay of a particle P, with mass m, into two lighter particles P_1 and P_2, with masses m_1 and m_2, respectively, where $m > m_1 + m_2$.

a) Write down the four-vector of each particle in the P rest-frame. Derive the energy of the final state particles P_1 and P_2, in the same reference frame, by using the conservation of four-momentum.

b) Consider now the muon decay $\mu(p) \to e(k)\nu(k_1)\bar{\nu}(k_2)$, where the four-momentum of each particle is denoted by the four-vectors inside the parentheses. The invariant mass of the neutrino pair is defined by $q^2 = (k_1 + k_2)^2$. Show that the electron energy E_e, in the muon rest frame, reads

$$E_e = \frac{m_\mu^2 + m_e^2 - q^2}{2m_\mu}, \tag{8}$$

where $m_{e(\mu)}$ is the electron (muon) mass. Derive the minimum and maximum values of E_e in this process under the assumption that neutrinos are massless.

Exercise 4: Mandelstam variables

Consider the scattering process $P_1 + P_2 \to P_3 + P_4$ where P_i are particles with mass m_i and four-momentum k_i (with $i = 1, 2, 3, 4$). Mandelstam variables are Lorentz-invariant quantities defined by

$$s = (k_1 + k_2)^2, \qquad t = (k_1 - k_3)^2, \qquad \text{and} \qquad u = (k_1 - k_4)^2. \tag{9}$$

a) Write down the products $k_i \cdot k_j$ as a function of s, t, u and m_i for the different values of $i, j \in \{1, 2, 3, 4\}$. Show that the center-of-mass energy is given by $E_{\text{cm}} = \sqrt{s}$.

b) Show that the Mandelstam variables satisfy the relation

$$s + t + u = \sum_{i=1}^{4} m_i^2. \tag{10}$$

c) Let us assume that the particles have the same mass (i.e. $m \equiv m_i \; \forall i$). Write the scattering angle θ between the ingoing and outgoing particles, in the center-of-mass frame (see figure below), as a function of s, t and m_i.