MMP II
Exercise Sheet 3

FS 19
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Exercise 1 [Definitions (2 points)]

Let $G \subset \mathbb{C}$ be open and $f : G \to \mathbb{C} : z \mapsto f(z)$ a function, while $z = x + iy$ and $x, y \in \mathbb{R}$.

(a) When is f complex differentiable at $z_0 \in \mathbb{C}$?

(b) When is f analytical (holomorphic)?

(c) f can be considered a function $f : G \to \mathbb{R}^2 : (x, y) \mapsto (u(x, y), v(x, y))$, where $u(x, y) = Re(f(z))$ and $v(x, y) = Im(f(z))$.
What is the relationship between complex derivation $\frac{\mathrm{d}f}{\mathrm{d}z}$ and partial derivations $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ and what equations do they satisfy?

Exercise 2 [Holomorphic functions (4 points)]

(a) Find the holomorphic function

$$w(z) = u(x, y) + iv(x, y)$$

with

$$\begin{aligned} \text{(i)} \quad u(x, y) &= 3x^2 y - y^3, \\ \text{(ii)} \quad v(x, y) &= e^{-x} \cos y. \end{aligned}$$

– please turn over –

(b) Which of the following functions are holomorphic on $\mathbb{C}$?

$$\begin{aligned} \text{(i)} \quad z &\to |z|, \\ \text{(ii)} \quad z &\to \bar{z}, \\ \text{(iii)} \quad z &\to \mathrm{e}^z, \\ \text{(iv)} \quad z &\to \cos(z). \end{aligned}$$

Exercise 3 [Complex functions (3 points)]
Prove the following identity for $a, b \in \mathbb{R}$:

$$\left(\frac{ia-1}{ia+1}\right)^{ib} = \exp\left(-2\,b \operatorname{arccot} a\right).$$

Exercise 4 [Complex exponential (2 points)]
Find the solutions of $e^z = w$ for

$$\begin{array}{llll} \text{a)} & w = -1, & \text{c)} & w = 4 - 4\,i, \\ \text{b)} & w = \frac{-1-i}{\sqrt{2}}, & \text{d)} & w = -3 + \sqrt{3}\,i. \end{array}$$

Exercise 5 [Complex trigonometric functions (4 points)]

(a) Find the null points of $\sin z$ and $\cosh z$.

(b) What are the solutions of $\cos z = i$?

(c) Find $\lim_{|z|\to\infty} \frac{\sin z}{z}$ for $z = x$, $z = iy$ and $z = x + iy$.