

**Exercise 1. Goldberger-Treiman relation**

The matrix element of the axial isospin current in the nucleon can be written in terms of form factors as

$$\langle N | J^{\mu 5a}(q) | N \rangle = \bar{u}(p') \left[ \gamma^\mu \gamma^5 F_1^5(q^2) + \frac{i\sigma^{\mu\nu} q_\nu}{2m} \gamma^5 F_2^5(q^2) + q^\mu \gamma^5 F_3^5(q^2) \right] \tau^a u(p), \quad (1)$$

with  $q = p - p'$  and  $\tau^a = \sigma^a/2$ .

1. In the limit of massless quarks, assume the conservation of the axial vector current and show that

$$g_A := F_1^5(0) = \lim_{q^2 \rightarrow 0} \frac{q^2}{2m_N} F_3^5(q^2), \quad (2)$$

where  $m_N$  is the mass of the nucleon.

2. Given that the low-energy pion-nucleon interaction is parametrised by the Lagrangian

$$\mathcal{L}_{\text{int}} = 2ig_{\pi NN} \pi^a \bar{N} \gamma^5 \tau^a N, \quad (3)$$

consider the contribution to the matrix element (1) where the interaction is mediated by the exchange of a massless pion. Show the Goldberger-Treiman relation

$$g_A = \frac{f_\pi}{m_N} g_{\pi NN}. \quad (4)$$

**Exercise 2. Gell-Mann–Okubo Mass Formula and Weinberg Ratio**

Consider the Lagrangian of chiral perturbation theory at order  $p^2$ ,

$$\mathcal{L}_{\text{CHPT}, p^2} = \frac{v^2}{4} \text{Tr} \left( D_\mu U D^\mu U^\dagger + \chi U^\dagger + \chi^\dagger U \right), \quad (5)$$

where  $U = \exp(i\sqrt{2}\Phi/v)$  is a  $SU(3)$  matrix with

$$\Phi = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -2\frac{\eta_8}{\sqrt{6}} \end{pmatrix}, \quad (6)$$

and  $\chi = 2B \text{diag}(m_u, m_d, m_s)$  is the quark mass matrix. Expand the Lagrangian in  $\Phi$  and compute the mass of the eight mesons. Verify the Gell-Mann–Okubo mass formula

$$4m_K^2 - 3m_\eta^2 - m_\pi^2 = 0 \quad (7)$$

and the Weinberg ratio of quark masses

$$\frac{2m_K^2 - m_\pi^2}{m_\pi^2} = \frac{2m_s}{m_d + m_u}, \quad (8)$$

where  $m_\pi^2$  and  $m_K^2$  are the average masses of the pion and the kaon, respectively.