Exercise 1. *Instantons*

In four-dimensional Euclidean space ($\mu, \nu = 1, 2, 3, 4$), consider the instanton configuration of a $SU(2)$ gauge field

$$\underline{A}^a_\mu = \frac{2}{g} \frac{\eta^a_{\mu\nu} x_\nu}{|x|^2 + \lambda^2}\,, \tag{1}$$

where λ is an arbitrary distance and η is 't Hooft's mixed colour and space-time tensor defined as

$$\eta^a_{\mu\nu} = \epsilon_{a\mu\nu 4} - \delta_{\mu 4}\delta_{a\nu} + \delta_{\nu 4}\delta_{a\mu} \tag{2}$$

or, equivalently,

$$\eta^a_{\mu\nu} = \begin{cases} \epsilon_{a\mu\nu} & \text{if } \mu \neq 4,\ \nu \neq 4 \\ \delta_{a\mu} & \text{if } \mu \neq 4,\ \nu = 4 \\ -\delta_{a\nu} & \text{if } \mu = 4,\ \nu \neq 4 \\ 0 & \text{if } \mu = \nu = 4\,. \end{cases} \tag{3}$$

The tensor η satisfies the contraction identities:

$$\begin{aligned} \eta^a_{\mu\nu} &= -\eta^a_{\nu\mu} \\ \epsilon_{abc}\eta^b_{\mu\nu}\eta^c_{\rho\sigma} &= \delta_{\mu\rho}\eta^a_{\nu\sigma} - \delta_{\mu\sigma}\eta^a_{\nu\rho} - \delta_{\nu\rho}\eta^a_{\mu\sigma} + \delta_{\nu\sigma}\eta^a_{\mu\rho} \\ \eta^a_{\mu\nu}\eta^a_{\mu\nu} &= 12\,. \end{aligned} \tag{4}$$

1. *Show that the field strength tensor $\underline{F}^a_{\mu\nu} = \partial_\mu \underline{A}^a_\nu - \partial_\nu \underline{A}^a_\mu + g\epsilon_{abc}\underline{A}^b_\mu \underline{A}^c_\nu$ is given by*

$$\underline{F}^a_{\mu\nu} = \frac{4}{g} \frac{-\lambda^2 \eta^a_{\mu\nu}}{(|x|^2 + \lambda^2)^2}\,. \tag{5}$$

2. *Compute the action corresponding to this field configuration,*

$$\underline{S} = \frac{1}{4} \int \mathrm{d}^4x \underline{F}^a_{\mu\nu} \underline{F}^a_{\mu\nu}\,. \tag{6}$$

Solution. The instanton solution discussed in the following was first reported in [BPST75]. The formulation that introduces $\eta^a_{\mu\nu}$ is from [tH76], where the quantum effects of instantons were considered.

1. From the expression of the gauge field $\underline{A}^a_\mu$ given in eq. (1), we have We have

$$\begin{aligned} \partial_\mu \underline{A}_\nu &= \frac{-4}{g} \frac{1}{(x^2 + \lambda^2)^2} x_\mu \eta^a_{\nu\kappa} x_\kappa - \frac{2}{g} \frac{1}{x^2 + \lambda^2} \eta^a_{\mu\nu}\,, \\ \partial_\nu \underline{A}_\mu &= \frac{-4}{g} \frac{1}{(x^2 + \lambda^2)^2} x_\nu \eta^a_{\mu\kappa} x_\kappa + \frac{2}{g} \frac{1}{x^2 + \lambda^2} \eta^a_{\mu\nu}\,, \end{aligned} \tag{S.1}$$

from which we derive

$$\partial_\mu \underline{A}_\nu - \partial_\nu \underline{A}_\mu = \frac{-4}{g} \frac{1}{(x^2 + \lambda^2)^2} x_\mu \eta^a_{\nu\kappa} x_\kappa + \frac{4}{g} \frac{1}{(x^2 + \lambda^2)^2} x_\nu \eta^a_{\mu\kappa} x_\kappa - \frac{4}{g} \frac{1}{x^2 + \lambda^2} \eta^a_{\mu\nu}. \tag{S.2}$$

For the non-Abelian term of the field strength,

$$g\epsilon_{abc}\underline{A}_\mu^b\underline{A}_\nu^c = \frac{4}{g}\frac{1}{(x^2+\lambda^2)^2}\epsilon_{abc}\eta^b_{\mu\kappa}\eta^c_{\nu\rho}x_\kappa x_\rho\,, \tag{S.3}$$

we use the contraction identities of eq. (4) in order to write

$$\begin{aligned}\epsilon_{abc}\eta^b_{\mu\kappa}\eta^c_{\nu\rho}x_\kappa x_\rho &= \left(\delta_{\mu\nu}\eta^a_{\kappa\rho} - \delta_{\mu\rho}\eta^a_{\kappa\nu} - \delta_{\kappa\nu}\eta^a_{\mu\rho} + \delta_{\kappa\rho}\eta^a_{\mu\nu}\right)x_\kappa x_\rho \\ &= x_\mu\eta^a_{\nu\kappa}x_\kappa - x_\nu\eta^a_{\mu\kappa}x_\kappa + (x^2+\lambda^2)\eta^a_{\mu\nu} - \lambda^2\eta^a_{\mu\nu}\end{aligned} \tag{S.4}$$

and arrive at

$$g\epsilon_{abc}\underline{A}_\mu^b\underline{A}_\nu^c = \frac{4}{g}\left[\frac{1}{(x^2+\lambda^2)^2}(x_\mu\eta^a_{\nu\kappa}x_\kappa - x_\nu\eta^a_{\mu\kappa}x_\kappa - \lambda^2\eta^a_{\mu\nu}) + \frac{\eta^a_{\mu\nu}}{(x^2+\lambda^2)}\right]. \tag{S.5}$$

In the sum of eq.s (S.2) and (S.5), all the terms $\sim (x^2+\lambda^2)^{-2}$ cancel out and we obtain

$$\underline{F}^a_{\mu\nu} = -\frac{4}{g}\frac{\lambda^2\eta^a_{\mu\nu}}{(x^2+\lambda^2)^2}. \tag{S.6}$$

2. We insert the field strength tensor given in eq. (S.6) into the definition of the action and, by using $\eta^a_{\mu\nu}\eta^a_{\mu\nu} = 12$ we arrive at

$$\underline{S} = \frac{1}{4}\int \mathrm{d}^4x\underline{F}^a_{\mu\nu}\underline{F}^a_{\mu\nu} = \frac{48\lambda^4}{g^2}\int \mathrm{d}^4x_E\frac{1}{(x^2+\lambda^2)^4} = \frac{48\lambda^4}{g^2}\frac{\pi^2}{6\lambda^4} = \frac{8\pi^2}{g^2}\,, \tag{S.7}$$

where the Euclidean four-dimensional integral has been obtained from the $d\to 4$ limit of the formula derived Exercise Sheet 1 (or, equivalently, by moving to spherical coordinates and by applying Cauchy theorem).
From the above result we immediately see that the path-integral weight $e^{-\underline{S}}$ of the instanton configuration has an essential singularity at $g = 0$, which makes the weak-coupling expansion of this expression useless.

References

[BPST75] *A. A. Belavin, A. M. Polyakov, A. S. Schwartz, and Yu. S. Tyupkin. Pseudoparticle solutions of the yang-mills equations.* Physics Letters B*, 59(1):85 – 87, 1975.*

[tH76] *G. 't Hooft. Computation of the quantum effects due to a four-dimensional pseudoparticle.* Phys. Rev. D*, 14(12):3432–3450, Dec 1976.*