Exercise 1. ***The soliton of $\lambda\phi^4$ theory in $1+1$ dimensions***

Consider the scalar field theory with Lagrangian

$$\mathscr{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - V(\phi), \qquad V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4\,, \tag{1}$$

in one dimension ($g_{\mu\nu} = \mathrm{diag}(+1,-1)$) with $m^2 < 0$.

1. Determine the constant solutions of the equations of motion and shift the potential in such a way that they have vanishing energy.
2. Find the static solutions of the equations of motion that interpolate between two constant solutions (these interpolating solutions are called solitons). Use the ansatz $\phi(x) = a\tanh(bx)$.
3. Calculate the energy of the static soliton (*Hint*: $\int_{-\infty}^{\infty} \mathrm{d}x \cosh^{-4}(x) = 4/3$).
4. Check that the current
$$J^\mu = \epsilon^{\mu\nu}\partial_\nu\phi \tag{2}$$
is conserved. What are the possible values of $\int \mathrm{d}x J^0$ for a solution of the equations of motion?