Feynman Integrals and Intersection Theory

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Based on:

- **Pierpaolo Mastrolia** and **Sebastian Mizera**
  Feynman integrals and Intersection theory
  JHEP 1902 (2019) 139
- **H.Frellesvig, F.Gasparotto, S.Laporta, M.Mandal, P. Mastrolia, L.M., S.Mizera**
  Decomposition of Feynman Integrals on the Maximal Cut by Intersection Numbers
  JHEP 1905 (2019) 153
- **H.Frellesvig, F.Gasparotto, M.Mandal, P. Mastrolia, L.M., S.Mizera**
  Vector Space of Feynman Integrals and Multivariate Intersection Numbers
  Phys.Rev.Lett. 123 (2019) no.20, 201602
- **H.Frellesvig, F.Gasparotto, S.Laporta, M.Mandal, P. Mastrolia, L.M., S.Mizera**
  Decomposition of Feynman Integrals by multivariate Intersection Numbers
  ArXiv:2008.04823

Motivation

Higgs discovery
Success of Standard Model

$\Downarrow$

SM as effective theory
(Dark Matter, Dark Energy...)

$\Downarrow$

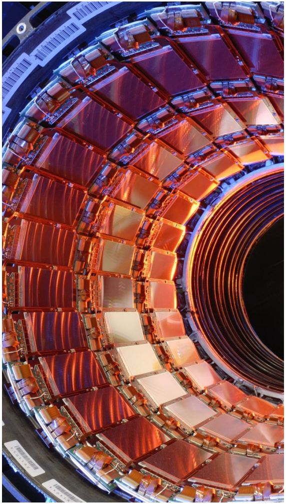
New physics hide in subtle effects

$\Downarrow$

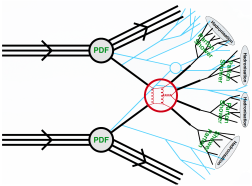
Precision calculations needed

$\Downarrow$

Scattering amplitudes are
powerful means to access it



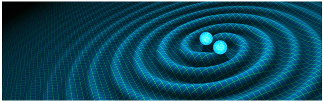
Motivation

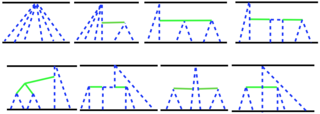


Scattering amplitudes are at the **core** of cross sections measured in colliders

very effective tool to know **gravitational waveforms** with high precision in weak field approximation

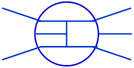
[Foffa,Sturani,Mastrolia,Sturm (2016)]
[Bern,Zeng et al. (2019)]
[Foffa,Sturani,Mastrolia,Sturm,Torres Bobadilla (2019)]...





Challanges

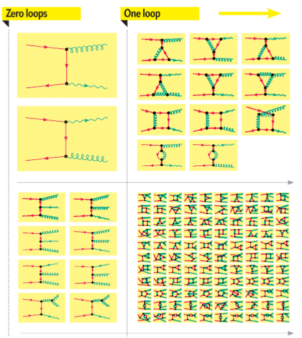
Scattering amplitudes are built out of many Feynman integrals:



$$I_{a_1,\ldots,a_N} = \int \prod_{i=1}^{L} d^d k_i \frac{1}{D_1^{a_1}\ldots D_N^{a_N}}$$

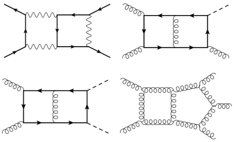
Higher precision $\Rightarrow$ Higher loop

Complexity of the calculations increases quickly



[Bern, Dixon, Kosower (2012)]

State of the art calculations at 2 loop, such as



requires $\mathcal{O}(10000)$ integrals. Needs to evaluate them all? No!

Integration By Parts Identities

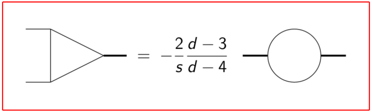
Linear relations among integrals: **Integration By Parts Identities - IBPs**

[Chetyrkin, Tkachov (1981)]

[Laporta (2001)]...

$$\int \prod_{i=1}^{L} d^d k_i \, \frac{\partial}{\partial k_j^{\mu}} \left(\frac{v^{\mu}}{D_1^{a_1} \ldots D_N^{a_N}} \right) = 0 \Rightarrow c_1 \, I_{a_1+1,\ldots,a_N} + \cdots + c_N I_{a_1,\ldots,a_N+1} = 0$$

Integrals related by a total derivative



Linear System ⇒ Gauss Elimination ⇒ Master Integrals $\{J_i\}$ - MIs

Decomposition of an Integral in terms of **MIs**

$$I_{a_1,\ldots,a_N} = \sum_{i=1}^{\nu} c_i J_i$$

Integration By Parts Identities

[Kotikov '91, Remiddi '97, Gehrmann & Remiddi '00, Argeri & Mastrolia '07, Henn '13]
[Tarasov '96, Lee '07]

MIs as solutions of

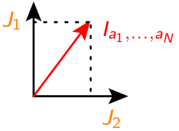
- Differential equations (DEQ) $\Rightarrow$ Built by means of **IBPs**
- Dimensional recurrence relations

$$\partial_s J_k = \sum_{i=1}^{\nu} c_{ki} J_i \quad \Rightarrow \quad \partial_s \mathbf{J} = \mathbb{A}\mathbf{J} \qquad \mathbb{A} = \begin{pmatrix} \blacksquare & & & \\ * & \blacksquare & \blacksquare & \\ * & \blacksquare & \blacksquare & \\ * & * & * & \blacksquare \end{pmatrix}$$

IBPs Drawbacks:

- # equations grows dramatically
- manipulation of large intermediate expressions

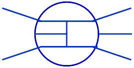
Possible bottleneck of multiloop calculations
Can we directly project integrals on MI?



Intersection theory

In **Baikov** representation

[Aomoto, Kita, Matsumoto, Mizera, ...]
[Mastrolia, Mizera (2018)]



$$I_{a_1,\ldots,a_N} = \int_{\mathcal{C}} u(z)\,\varphi(z)$$

$$u(z) = \prod_i P_i(z)^{\gamma_i} \;\Rightarrow\; u(z) \text{ multivalued function s.t. } P_i\left(\partial\mathcal{C}\right) = 0$$

$$\varphi(z) = \hat{\varphi}(z)\,\mathrm{d}z \Rightarrow \varphi(z) \text{ single valued form}$$

Total derivative translates to

$$\int_{\mathcal{C}} \mathrm{d}(u\,\varphi) = \int_{\mathcal{C}} \mathrm{d}u\,\varphi + u\,\mathrm{d}\varphi = \int_{\mathcal{C}} u\left(\frac{\mathrm{d}u}{u} + \mathrm{d}\right)\varphi$$

$$= \int_{\mathcal{C}} u\,(\omega + \mathrm{d})\,\varphi = \int_{\mathcal{C}} u\,\nabla_\omega\varphi = 0 \quad \mathcal{P}_\omega = \{z \,|\, z \text{ is a pole of } \omega\}$$

rewriting **Integration by Parts Identities** as

$$\int_{\mathcal{C}} u\,(\varphi + \nabla_\omega\xi) = \int_{\mathcal{C}} u\,\varphi \quad \Rightarrow \quad \varphi \sim \varphi + \nabla_\omega\xi$$

Intersection theory

Equivalence class between forms defines the Twisted cohomology group.

$$ {}_\omega\langle\varphi| \equiv \{\varphi|\nabla_\omega\varphi = 0\}/\{\nabla_\omega\xi\} = H^n_\omega $$

Key relation between IBPs and Twisted Cohomology

[Aomoto (1975)]
[Lee, Pomeransky (2013)]

$$ \begin{aligned} \nu &= dim\,(H^n_\omega) \\ &= \chi(X) = (-1)^n\,(n+1-\chi(\mathcal{P}_\omega)\,) \\ &= \{\#\text{ of solutions of }\omega = 0\} \end{aligned} $$

Contours have similar structure

$$ \int_{\mathcal{C}} u\varphi = \int_{\mathcal{C}+\partial_\omega g} u\varphi \quad \Rightarrow \quad |\mathcal{C}] = H_n^\omega \text{Twisted Homology group} $$

Feynman integrals are **pairing**

$$ \langle\varphi|\mathcal{C}] = \underbrace{\int_{\mathcal{C}}}_{cycle} u(z)\,\overbrace{\varphi(z)}^{cocycle} $$

Dual integrals

$$ [\mathcal{C}|\varphi\rangle = \int_{\mathcal{C}} u^{-1}(z)\,\varphi(z) = \int_{\mathcal{C}} u^{-1}\,(\varphi + \nabla_{-\omega}\xi) $$

$$ \Rightarrow |\varphi\rangle = H^n_{-\omega} \quad , \quad [C| = H_n^{-\omega} $$

Twisted intersection number

Seeking relations between integrals (forms) $\Rightarrow$ **Intersection number**

$$\langle \varphi_L | \varphi_R \rangle = \frac{1}{(2\pi i)^n} \int \iota(\varphi_L) \wedge \varphi_R$$

[Mastrolia, Mizera (2018)]
[Frellesvig, Gasparotto, Laporta, Mandal, Mastrolia, **L.M.**, Mizera (2019)]

IBP built in naturally within such formalism.

$$\langle \varphi_L + \nabla_\omega \xi | \varphi_R \rangle = \langle \varphi_L | \varphi_R + \nabla_{-\omega} \xi \rangle = \langle \varphi_L | \varphi_R \rangle$$

Define basis of independent form and dual form

$$\langle e_1|, \quad \langle e_2|, \quad \cdots, \quad \langle e_\nu| \quad , \quad |h_1\rangle, \quad |h_2\rangle, \quad \cdots, \quad |h_\nu\rangle$$

Build the matrix

$$\mathbf{M} = \begin{pmatrix} \langle \varphi_L | \varphi_R \rangle & \langle \varphi_L | h_1 \rangle & \langle \varphi_L | h_2 \rangle & \dots & \langle \varphi_L | h_\nu \rangle \\ \langle e_1 | \varphi_R \rangle & \langle e_1 | h_1 \rangle & \langle e_1 | h_2 \rangle & \dots & \langle e_1 | h_\nu \rangle \\ \langle e_2 | \varphi_R \rangle & \langle e_2 | h_1 \rangle & \langle e_2 | h_2 \rangle & \dots & \langle e_2 | h_\nu \rangle \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \langle e_\nu | \varphi_R \rangle & \langle e_\nu | h_1 \rangle & \langle e_\nu | h_2 \rangle & \dots & \langle e_\nu | h_\nu \rangle \end{pmatrix} \equiv \begin{pmatrix} \langle \varphi_L | \varphi_R \rangle & \mathbf{A}^\intercal \\ \mathbf{B} & \mathbf{C} \end{pmatrix}$$

Master Decomposition Formula

$\langle \varphi_L |$ **depends** on the basis element

$$\det \mathbf{M} = \det \mathbf{C} \left(\langle \varphi_L | \varphi_R \rangle - \mathbf{A}^\intercal\, \mathbf{C}^{-1}\, \mathbf{B} \right) = 0$$

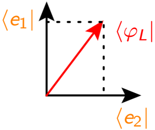
$$\Downarrow$$

$$\langle \varphi_L | \varphi_R \rangle = \mathbf{A}^\intercal\, \mathbf{C}^{-1}\, \mathbf{B} = \sum_{i,j=1}^{\nu} \langle \varphi_L | h_j \rangle \, (\mathbf{C}^{-1})_{ji} \, \langle e_i | \varphi_R \rangle$$

Since $| \varphi_R \rangle$ is arbitrary

$$\langle \varphi_L | = \sum_{i,j=1}^{\nu} \langle \varphi_L | h_j \rangle \, \left(\mathbf{C}^{-1} \right)_{ji} \langle e_i | \overset{\mathbf{C}_{ij} = \delta_{ij}}{=} \sum_{i=1}^{\nu} \langle \varphi_L | h_i \rangle \, \langle e_i | = \sum_{i=1}^{\nu} c_i \, J_i$$

Direct projection



Intersection number

$$\langle \varphi_L | \varphi_R \rangle = ?$$

Natural pairing $\to$ integration $\to$ ill defined

$$\int_X \varphi_L \wedge \varphi_R \quad \Rightarrow \quad \begin{array}{l} \varphi_L \wedge \varphi_R = 0 \\ X \text{ non compact} \end{array}$$

mandatory to regularize it without changing the valure of the intersection number

$$\varphi_L \Rightarrow \iota(\varphi_L) = \varphi_L + \nabla_\omega \xi$$

$$\nabla_\omega \xi \sim \sum_{z_i \in \mathcal{P}_\omega} \Theta(\epsilon - |z - z_i|^2)\, \varphi_L$$

so that $\iota(\varphi_L)$ has compact support

$$\nabla_\omega \xi \sim \nabla_\omega^{-1} \varphi_L$$

so that it is non - zero

$$\xi = \sum_{z_i \in \mathcal{P}_\omega} \Theta(\epsilon - |z - z_i|^2) \nabla_\omega^{-1} \varphi_L$$

Intersection number

$$\iota(\varphi) = \left(1 - \sum_{z_i \in \mathcal{P}_\omega} \Theta(\epsilon - |z - z_i|^2)\right) \varphi_L + \sum_{z_i \in \mathcal{P}_\omega} \delta(\epsilon - |z - z_i|^2) \nabla_\omega^{-1} \varphi_L$$

With the regularization

$$\langle \varphi_L | \varphi_R \rangle = \frac{1}{2\pi i} \int \iota(\varphi_L) \wedge \varphi_R$$

$$= \frac{1}{2\pi i} \int \sum_{z_i \in \mathcal{P}_\omega} \delta(\epsilon - |z - z_i|^2) \left(\nabla_\omega^{-1} \varphi_L\right) \wedge \varphi_R$$

$$\overset{\text{univ.}}{=} \sum_{p \in \mathcal{P}} \text{Res}\left(\psi_p \varphi_R\right) \qquad \nabla_\omega \psi_p = \varphi_L$$

Univariate computation

$$\langle \varphi_L | \varphi_R \rangle = \frac{1}{(2\pi i)} \int \iota(\varphi_L) \wedge \varphi_R = \sum_{p \in \mathcal{P}_\omega} \underset{z=p}{Res} \left(\left(\nabla_\omega^{-1} \varphi_L \right) \varphi_R \right) = - \sum_{p \in \mathcal{P}_\omega} \underset{z=p}{Res} \left(\varphi_L \nabla_{-\omega}^{-1} \varphi_R \right)$$

$$= \sum_{p \in \mathcal{P}_\omega} \underset{z=p}{Res} \left(\psi_p \, \varphi_R \right) \overset{dLog}{=} \sum_{p \in \mathcal{P}_\omega} \frac{Res_{z=p} \left(\varphi_L \right) \; Res_{z=p} \left(\varphi_R \right)}{Res_{z=p} \left(\omega \right)}$$

$$\psi_p = \nabla_\omega^{-1} \varphi_L \quad \Rightarrow \quad (\mathrm{d} + \omega) \; \psi_p = \varphi_L$$

only **local** solution to ψ_p needed $\Rightarrow$ power series ansatz

$$\psi_p = \sum_{j=\min}^{\max} \psi_p^{(j)} \tau^j + \mathcal{O} \left(\tau^{\max+1} \right)$$

ψ_p obtained by **pattern matching**

Sanity check:

$$\langle \nabla_\omega \xi | \varphi_R \rangle = \sum_{p \in \mathcal{P}_\omega} \underset{z=p}{Res} \left(\xi \varphi_R \right)$$

Univariate computation

$$\langle \varphi_L | \varphi_R \rangle = \frac{1}{(2\pi i)} \int \iota(\varphi_L) \wedge \varphi_R = \sum_{p \in \mathcal{P}_\omega} \underset{z=p}{Res} \left(\left(\nabla_\omega^{-1} \varphi_L \right) \varphi_R \right) = - \sum_{p \in \mathcal{P}_\omega} \underset{z=p}{Res} \left(\varphi_L \nabla_{-\omega}^{-1} \varphi_R \right)$$

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Sanity check:

$$\langle \nabla_\omega \xi | \varphi_R \rangle = \sum_{p \in \mathcal{P}_{\xi \varphi_R}} \underset{z=p}{Res} \left(\xi \varphi_R \right)$$

Univariate computation

$$\langle\varphi_L|\varphi_R\rangle = \frac{1}{(2\pi i)}\int \iota(\varphi_L)\wedge\varphi_R = \sum_{p\in\mathcal{P}_\omega} \underset{z=p}{Res}\left(\left(\nabla_\omega^{-1}\varphi_L\right)\varphi_R\right) = -\sum_{p\in\mathcal{P}_\omega} \underset{z=p}{Res}\left(\varphi_L\nabla_{-\omega}^{-1}\varphi_R\right)$$

$$= \sum_{p\in\mathcal{P}_\omega} \underset{z=p}{Res}\left(\psi_p\,\varphi_R\right) \overset{dLog}{=} \sum_{p\in\mathcal{P}_\omega} \frac{Res_{z=p}\left(\varphi_L\right)\ Res_{z=p}\left(\varphi_R\right)}{Res_{z=p}\left(\omega\right)}$$

$$\psi_p = \nabla_\omega^{-1}\varphi_L \quad\Rightarrow\quad (\mathrm{d}+\omega)\,\psi_p = \varphi_L$$

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ψ_p obtained by **pattern matching**

Sanity check:

$$\langle\nabla_\omega\xi|\varphi_R\rangle = \sum_{p\in\mathcal{P}_{\xi\varphi_R}} \underset{z=p}{Res}\left(\xi\varphi_R\right) = 0$$

Reduction on the Maximal Cut

Maximal Cut $\Rightarrow$ univariate integral representation

$$u = \left(\frac{1}{4} z^2 (s - 2z - 1)(s - 2z + 3)\right)^{\frac{d-5}{2}}$$

$\omega = d \log u = 0$ 2 sols. $\Rightarrow$ 2 MIs $\nu = 2$

The MIs chosen as

$$J_1 = I_{1,1,1,1,1,1,1,1,1;0} = \langle e_1 | \mathcal{C}] = \langle 1 | \mathcal{C}] \quad \& \quad J_2 = I_{1,1,1,1,1,1,1,1,1;-1} = \langle e_2 | \mathcal{C}] = \langle z | \mathcal{C}]$$

Decompose

$$I_{1,1,1,1,1,1,1,1,1;-2} = \langle \varphi | \mathcal{C}] = \langle z^2 | \mathcal{C}]$$

Compute

$$\langle \varphi | e_i \rangle \quad i = 1,2 \quad \& \quad \mathbf{C}_{ij} = \langle e_i | e_j \rangle \quad i,j = 1,2$$

Plug in the **Master Decomposition Formula**

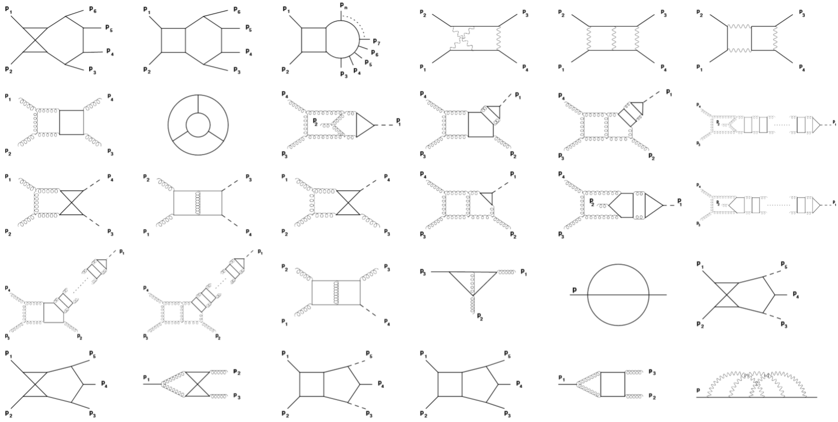
$$c_i = \sum_{j=1}^{2} \langle \varphi | e_j \rangle \left(\mathbf{C}^{-1}\right)_{ji} \quad c_1 = -\frac{(d-4)(s-1)(s+3)}{4(2d-7)} \quad c_2 = \frac{(3d-11)(s+1)}{2(2d-7)}$$

agreement with SYS

Univariate Examples

$\mathcal{O}(30)$ examples checked on the maximal cut

[Frellesvig, Gasparotto, Laporta, Mandal, Mastrolia, **L.M.**, Mizera (2019)]



Multivariate :: Fibration

Univariate: **known**; Multivariate?

$$\int u(z)\varphi_L(z) \quad \overset{?}{\Leftarrow} \quad \int \mathrm{d}z_n \int \mathrm{d}z_{n-1} \cdots \int \mathrm{d}z_1\, u(z_1,\ldots,z_n)\,\varphi_L(z_1,\ldots,z_n)$$

$$\int \mathrm{d}z_n\, \varphi_L^{(n)}(z_n)\, u^{(n)}(z_n)$$

Integration connects them. How does it translate to our formalism?

$$\int u(\mathbf{z})\varphi_L(\mathbf{z}) = \int \mathrm{d}z_n \cdots \int \mathrm{d}z_2 \underbrace{\int \mathrm{d}z_1\, u(z_1,\ldots,z_n)\,\hat{\varphi}_L(z_1,\ldots,z_n)}_{\exists\, \nu_1 \text{ MI in } z_1}$$

$$= \int \mathrm{d}z_n \cdots \int \mathrm{d}z_3 \underbrace{\int \mathrm{d}z_2 \sum_i c_i^{(1)}(z_n,\ldots,z_2)\, J_i^{(1)}(z_n,\ldots,z_2)}_{\exists\, \nu_2 \text{ MI in } z_2}$$

$$\vdots$$

$$= \int \mathrm{d}z_n \sum_i c_i^{(n)}(z_n) J_i^{(n)}(z_n)$$

Multivariate :: Fibration

$$\int u(\mathbf{z})\varphi_L(\mathbf{z}) = \int \mathrm{d}z_n \sum_i c_i^{(n)}(z_n) J_i^{(n)}(z_n) = \int \mathrm{d}z_n \sum_i \varphi_{L,i}^{(n)}(z_n) \int u(\mathbf{z})\, e_i^{(\mathbf{n-1})}(\mathbf{z})$$

[Mizera (2019)]
[Frellesvig, Gasparotto, Mandal, Mastrolia, **L.M.**, Mizera (2019)]

$$\Downarrow$$

$$\langle \varphi_L | = \sum_{i=1}^{\nu_{\mathbf{n-1}}} \langle \varphi_{L,i}^{(n)} | \wedge \langle e_i^{(\mathbf{n-1})} |$$

$\varphi_{L,i}^{(n)}$ **coefficient** of the reduction

$$\langle \varphi_{L,i}^{(n)} | = \sum_j \langle \varphi_L | h_j^{(\mathbf{n-1})} \rangle \left(\mathbf{C}_{(\mathbf{n-1})}^{-1} \right)_{ji}$$

Same decomposition on the **dual** basis

$$|\varphi_R\rangle = \sum_{i=1}^{\nu_{\mathbf{n-1}}} |\varphi_{R,i}^{(n)}\rangle \wedge |h_i^{(\mathbf{n-1})}\rangle \quad \Rightarrow \quad |\varphi_{R,i}^{(n)}\rangle = \sum_j \left(\mathbf{C}_{(\mathbf{n-1})}^{-1} \right)_{ij} \langle e_j^{(\mathbf{n-1})} | \varphi_R \rangle$$

Multivariate :: Computation

[Mizera (2019)]
[Frellesvig, Gasparotto, Mandal, Mastrolia, **L.M.**, Mizera (2019)]

We are interested in

$$
\begin{aligned}
\langle \varphi_L | \varphi_R \rangle &= \frac{1}{(2\pi i)^2} \int_X \iota(\varphi_L) \wedge \varphi_R \\
&= \frac{1}{(2\pi i)} \int_{X_2} \left(\iota(\varphi^{(2)}_{L,i}) \wedge \varphi^{(2)}_{R,j} \; \overbrace{\frac{1}{(2\pi i)} \int_{X_1} \iota(e^{(1)}_i) \wedge h^{(1)}_j}^{\langle e^{(1)}_i | h^{(1)}_j \rangle = \mathbf{C}^{(1)}_{ij}} \right) \\
&= \frac{1}{(2\pi i)} \int_{X_2} \left(\iota(\varphi^{(2)}_{L,i}) \wedge \varphi^{(2)}_{R,j} \; \mathbf{C}^{(1)}_{ij} \right) \\
&= \sum_{p \in \mathcal{P}_{\Omega^{(2)}}} \underset{z_2 = p}{Res} \left(\psi^{(2)}_{p,i} \; \hat{\varphi}^{(2)}_{R,j} \; \mathbf{C}^{(1)}_{ij} \right)
\end{aligned}
$$

$$
\Rightarrow \quad \frac{1}{(2\pi i)^n} \int \iota(\varphi_L) \wedge \varphi_R = \sum_{p \in \mathcal{P}_{\Omega^{(n)}}} \underset{z_n = p}{Res} \left(\psi^{(n)}_{p,i} \; \varphi^{(n)}_{R,j} \; \mathbf{C}^{(\mathbf{n-1})}_{ij} \right)
$$

$\psi^{(n)}_i$ **generalization** of univariate case

$$
\partial_{z_n} \psi^{(n)}_i + \psi^{(n)}_j \hat{\Omega}^{(n)}_{ji} = \hat{\varphi}^{(n)}_{L,i}
$$

Multivariate :: Connection

New connection arise: $\Omega_{ij}^{(n)}$. In the 2 variables case:

$$\int \varphi(z_1,z_2)\, u(z_1,z_2) = \sum_i^{\nu_1} \int \mathrm{d}z_2 \varphi_i^{(2)}(z_2) \int \mathrm{d}z_1\, e_i^{(1)}(z_1,z_2)\, u(z_1,z_2)$$

$$= \sum_i^{\nu_1} \int \mathrm{d}z_2 \varphi_i^{(2)}(z_2) u_i(z_2)$$

total derivative is

$$0 = \sum_i^{\nu_1} \int \mathrm{d}\left(\varphi_i^{(2)}(z_2) u_i(z_2)\right) = \sum_i^{\nu_1} \int \left(\mathrm{d}\varphi_i^{(2)}(z_2) u_i(z_2) + \varphi_i^{(2)}(z_2)\, \mathrm{d}u_i(z_2)\right)$$

$$= \sum_i^{\nu_1} \int \left(\mathrm{d}\varphi_i^{(2)}(z_2)\delta_{ij} + \varphi_i^{(2)}(z_2)\, \Omega_{ij}^{(2)}(z_2)\right) u_j(z_2)$$

Multivariate :: Connection

$$\begin{aligned}
\mathrm{d}u_i(z_2) &= \mathrm{d}_{z_2} \int e_i^{(1)}(z_1, z_2) u(z_1, z_2) \\
&= \int \left(\mathrm{d}_{z_2} e_i^{(1)}(z_1, z_2) + \frac{\mathrm{d}_{z_2} u(z_1, z_2)}{u(z_1, z_2)} \wedge e_i^{(1)}(z_1, z_2) \right) u(z_1, z_2) \\
&= \int u(z_1, z_2)(\mathrm{d}_{z_2} + \omega_2 \wedge) e_i^{(1)}(z_1, z_2) \Rightarrow \Omega_{ij}^{(2)}(z_2) \int e_j^{(1)}(z_1, z_2) u(z_1, z_2)
\end{aligned}$$

$\Omega_{ij}^{(2)}(z_2)$ **projection** of the outer connection

$$\Omega_{ij}^{(2)}(z_2) = \sum_k \langle (\mathrm{d}_{z_2} + \omega_2 \wedge)\, e_i^{(1)} | h_k^{(1)} \rangle \, \left(\mathbf{C}_{(1)}^{-1} \right)_{kj}$$

$$\Downarrow$$

$$\hat{\Omega}_{ij}^{(n)}(z_n) = \sum_k \langle (\mathrm{d}_{z_n} + \omega_n \wedge)\, e_i^{(\mathbf{n-1})} | h_k^{(\mathbf{n-1})} \rangle \, \left(\mathbf{C}_{(\mathbf{n-1})}^{-1} \right)_{kj}$$

Multivariate :: The algorithm

Goal: $\langle \varphi_L | \varphi_R \rangle$

Inputs

$\langle \varphi_L | \, , \, | \varphi_R \rangle$ n-forms,

$\omega = \sum_i^n \omega_i$ connection,

ν_{n-1} number master (n-1)-forms

$\langle e_i^{(\mathbf{n-1})} | \, , \, | h_j^{(\mathbf{n-1})} \rangle$ inner basis

$\Rightarrow$

Get

$\mathbf{C}_{ij}^{(\mathbf{n-1})} = \langle e_i^{(\mathbf{n-1})} | h_j^{(\mathbf{n-1})} \rangle$ metric matrix,

$\langle \varphi_{L,i}^{(n)} | \, , \, | \varphi_{R,i}^{(n)} \rangle$ projected form,

$\Omega_{ij}^{(n)}$ projected connection

Compute

see also [Matsumoto (1998)] [Matsubara et al. (2019)] [Weinzierl (2020)]

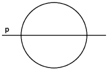
$$\langle \varphi_L | \varphi_R \rangle = \sum_{p \in \mathcal{P}_{\Omega^{(n)}}} \underset{z_n = p}{Res} \left(\psi_{p,i}^{(n)} \, \varphi_{R,j}^{(n)} \, \mathbf{C}_{ij}^{(\mathbf{n-1})} \right)$$

$$\partial_{z_n} \psi_i^{(n)} + \hat{\Omega}_{ij}^{(n)} \psi_j^{(n)} = \hat{\varphi}_{L,i}^{(n)}$$

Terminating conditions: $\Omega^{(1)} = \omega_1, \; \mathbf{C}^{(0)} = \mathbf{1}, \; \varphi_{L,R}^{(1)} = \varphi_{L,R}$

Multivariate :: Sunrise

Maximal Cut $\Rightarrow$ 2 variables integral representation



$$u = (z_1 z_2 (1 - z_1 - z_2))^{\gamma}$$

Decompose $I_{1,1,1,0,-1} = \langle \varphi | \mathcal{C}] = \langle z_2 | \mathcal{C}]$ on MIs $J_1 = I_{1,1,1,0,0} = \langle e^{(12)} | \mathcal{C}] = \langle 1 | \mathcal{C}]$

Compute $\langle z_2 | 1 \rangle$ & $\mathbf{C} = \langle 1 | 1 \rangle$

Inputs

$$\langle \varphi_L | = \langle 1 | \ , \ | \varphi_R \rangle = | 1 \rangle \ ,$$

$$\omega = \sum_i^n \omega_i \ , \ \nu_1 = 1$$

$$\langle e^{(1)} | = \langle z_1 | \ , \ | h^{(1)} \rangle = | z_1 \rangle$$

$\Rightarrow$

Get

$$\mathbf{C}^{(1)} = \frac{\gamma (z_2 - 1)^4}{8(2\gamma - 1)(2\gamma + 1)},$$

$$\langle \varphi_L^{(2)} | = \langle -\frac{2}{z_2 - 1} | \ , \ | \varphi_R^{(2)} \rangle = | -\frac{2}{z_2 - 1} \rangle$$

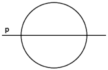
$$\hat{\Omega}^{(2)} = \frac{(3\gamma + 2) z_2 - \gamma}{(z_2 - 1) z_2}$$

Compute

$$\langle 1 | 1 \rangle = \frac{\gamma^2}{3(3\gamma - 2)(3\gamma - 1)(3\gamma + 1)(3\gamma + 2)}$$

Multivariate :: Sunrise

Maximal Cut $\Rightarrow$ 2 variables integral representation



$$u = (z_1 z_2(1 - z_1 - z_2))^{\gamma}$$

Decompose $I_{1,1,1,0,-1} = \langle \varphi | \mathcal{C}] = \langle z_2 | \mathcal{C}]$ on MIs $J_1 = I_{1,1,1,0,0} = \langle e^{(12)} | \mathcal{C}] = \langle 1 | \mathcal{C}]$

Compute $\langle z_2 | 1 \rangle$ & $\mathbf{C} = \langle 1 | 1 \rangle$

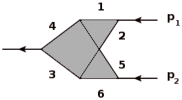
$$\langle z_2 | 1 \rangle = \frac{\gamma^2}{9(3\gamma - 2)(3\gamma - 1)(3\gamma + 1)(3\gamma + 2)} \quad \langle 1 | 1 \rangle = \frac{\gamma^2}{3(3\gamma - 2)(3\gamma - 1)(3\gamma + 1)(3\gamma + 2)}$$

Plug in the **Master Decomposition Formula**

$$c_i = \sum_{i=1}^{\nu} \langle \varphi | e_j \rangle \left(\mathbf{C}^{-1}\right)_{ji} = \frac{\langle z_2 | 1 \rangle}{\langle 1 | 1 \rangle} \quad \Rightarrow \quad c = \frac{1}{3}$$

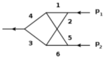
Multivariate :: Non Planar Triangle

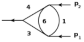
[Frellesvig, Gasparotto, Laporta, Mandal, Mastrolia, **L.M.**, Mizera (2020)]

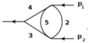


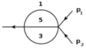
$$= \int \frac{u(\mathbf{z})\, f(\mathbf{z})}{z_1 z_2 z_3 z_4 z_5 z_6} \mathrm{d}^7 z$$

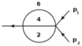
5 MIs
$$\begin{cases} \nu_{(1,2,3,4,5,6)} = 1 \\ \nu_{(1,3,4,6)} = 1 \\ \nu_{(2,3,4,6)} = 1 \\ \nu_{(2,4,6)} = 1 \\ \nu_{(1,3,5)} = 1 \end{cases}$$

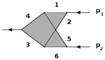
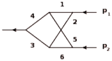
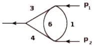
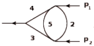
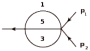
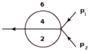
$J_1 =$ 

$J_2 =$ 

$J_3 =$ 

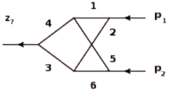
$J_4 =$ 

$J_5 =$ 

 $= c_1$  $+ c_2$  $+ c_3$  $+ c_4$  $+ c_5$ 

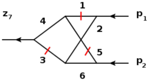
Multivariate :: Non Planar Triangle

[Frellesvig, Gasparotto, Laporta, Mandal, Mastrolia, **L.M.**, Mizera (2020)]



$$= \int \frac{u(\mathbf{z})\, z_7}{z_1 z_2 z_3 z_4 z_5 z_6} \mathrm{d}^7 z$$

Cut$_{\{1,3,5\}}$



$$= \int u_{135} \varphi_{135}$$

$$u_{135} = z_2^{\rho_2} z_4^{\rho_4} z_6^{\rho_6} u(0, z_2, 0, z_4, 0, z_6, z_7)$$

$$\hat{\varphi}_{135} = \frac{z_7}{z_2 z_4 z_6}$$

$$\varphi_{135} = \hat{\varphi}_{135} \mathrm{d}z_2 \wedge \mathrm{d}z_4 \wedge \mathrm{d}z_6 \wedge \mathrm{d}z_7$$

Number of MIs

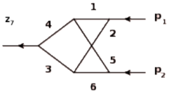
$$\nu_{(2467)} = 2$$
$$\nu_{(246)} = 3$$
$$\nu_{(24)} = 2$$
$$\nu_{(2)} = 2$$

Inner basis

$$\hat{e}_1^{(2467)} = \frac{1}{z_2 z_4 z_6} \quad \hat{e}_2^{(2467)} = 1$$
$$\hat{e}_1^{(246)} = z_6 \quad \hat{e}_2^{(246)} = z_4 \quad \hat{e}_3^{(246)} = z_2$$
$$\hat{e}_1^{(24)} = z_4 \quad \hat{e}_2^{(24)} = z_2$$
$$\hat{e}_1^{(2)} = 1 \quad \hat{e}_2^{(2)} = z_2$$

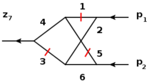
Multivariate :: Non Planar Triangle

[Frellesvig, Gasparotto, Laporta, Mandal, Mastrolia, **L.M.**, Mizera (2020)]



$$= \int \frac{u(\mathbf{z})\, z_7}{z_1 z_2 z_3 z_4 z_5 z_6} \mathrm{d}^7 z$$

Cut$_{\{1,3,5\}}$

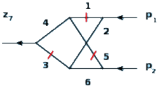


$$= \int u_{135} \varphi_{135}$$

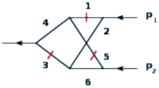
$$u_{135} = z_2^{\rho_2} z_4^{\rho_4} z_6^{\rho_6} u(0, z_2, 0, z_4, 0, z_6, z_7)$$

$$\hat{\varphi}_{135} = \frac{z_7}{z_2 z_4 z_6}$$

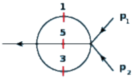
$$\varphi_{135} = \hat{\varphi}_{135} \mathrm{d}z_2 \wedge \mathrm{d}z_4 \wedge \mathrm{d}z_6 \wedge \mathrm{d}z_7$$



$= c_1$



$+ c_4$

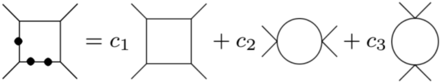


$$c_i = \sum_{j=1}^{2} \langle \varphi_{135} | h_j^{(2467)} \rangle \left(\mathbf{C}_{(2467)}^{-1} \right)_{ji} \quad \Rightarrow \quad c_1 = -\frac{s}{2} \quad , \quad c_2 = \frac{(d-3)(3d-10)(3d-8)}{2(d-4)^3 s^2}$$

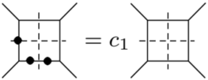
Top-down approach

[Frellesvig, Gasparotto, Laporta, Mandal, Mastrolia, **L.M.**, Mizera (2020)]

Consider

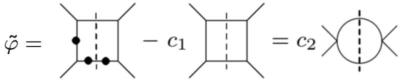


Obtain the first coefficient



$$\Rightarrow \qquad c_1 = \frac{-(d-7)(d-6)(d-5)}{2s^2 t}$$

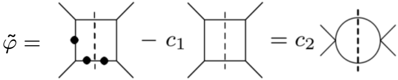
Use it to ease the decomposition:



$$c_2 \stackrel{?}{=} \frac{\langle \tilde{\varphi} | 1 \rangle}{\langle 1 | 1 \rangle}$$

Top-down approach

[Frellesvig, Gasparotto, Laporta, Mandal, Mastrolia, **L.M.**, Mizera (2020)]



$\tilde{\varphi}$ has spourious poles

$$\tilde{\varphi} = \varphi - \frac{c_1}{x_2 x_4} = \frac{P(x_2, x_4)}{st^2 x_2^2 x_4 \; B|^2_{x_1=x_3=0}}$$

$\tilde{\varphi}$ in an equivalence classe as one without poles $\phi \quad \Rightarrow \quad \phi = \tilde{\varphi} - \nabla_\omega \xi$

$$\xi = \frac{\sum_{i=-1,j=-1}^{2,2} \kappa_{1ij} x_2^i x_4^j \mathrm{d}x_2 + \sum_{i=-2,j=0}^{2,2} \kappa_{2ij} x_2^i x_4^j \mathrm{d}x_4}{B|_{x_1=x_3=0}}$$

Fit an ansatz so that all spourious poles vanishes

$$\phi = \frac{\tilde{P}(x_2, x_4)}{B|^2_{x_1=x_3=0}} \quad \Rightarrow \quad \text{No poles not in } \omega$$

$$c_2 = \frac{\langle \phi | 1 \rangle}{\langle 1 | 1 \rangle} = \frac{2(d-7)(d-5)(d-3)}{s^4 t}$$

Differential Equations

[Mastrolia, Mizera (2018)]
[Frellesvig, Gasparotto, Laporta, Mandal, Mastrolia, **L.M.**, Mizera (2019)]
[Mizera, Prokraka (2019)]
[Chen, Xu, Yang (2020)]

Intersection theory is a very **flexible** tool

$$\partial_s J_k = \sum_{i=1}^{\nu} c_{ki} J_i \qquad \Rightarrow \qquad \partial_s \boldsymbol{J} = \mathbb{A} \boldsymbol{J} \qquad \mathbb{A} = \begin{pmatrix} \blacksquare & & & \\ * & \blacksquare & \blacksquare & \\ * & \blacksquare & \blacksquare & \\ * & * & * & \blacksquare \end{pmatrix}$$

Derivative w.r.t. s translates to

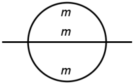
$$\int_{\mathcal{C}} \partial_s (u\,\varphi) = \int_{\mathcal{C}} \partial_s u\,\varphi + u\,\partial_s \varphi = \int_{\mathcal{C}} u \left(\frac{\partial_s u}{u} + \partial_s \right) \varphi$$
$$= \int_{\mathcal{C}} u\,(\sigma + \partial_s)\,\varphi$$

Obtain system of DEQ through intersection theory

$$\partial_s \langle e_i | = \langle (\partial_s + \sigma \wedge) e_i | = \langle \Phi_i |$$
$$= \langle \Phi_i | h_k \rangle \left(\boldsymbol{C}^{-1} \right)_{kj} \langle e_j | = \Omega_{ij} \langle e_j |$$

Differential Equations

Considering $m = 1, x = \sqrt{s}$ on maximal cut we have [Frellesvig, Gasparotto, Laporta, Mandal, Mastrolia, **L.M.**, Mizera (2019)]



$$u = z^{\frac{1}{2}} \left((z-4)(z-(x+1)^2)(z-(x-1)^2) \right)^{\frac{d-3}{2}}$$

$$\sigma = \frac{2(d-3)x(x^2-z-1)}{(x^4-2x^2(z+1)+(z-1)^2} \qquad \omega = d\log u = 0 \qquad 3 \text{ sols.} \Rightarrow 3 \text{ MIs} \qquad \nu = 3$$

The MIs chosen as

$$\varphi_1 = \frac{\mathrm{d}z}{z-(x+1)^2}, \qquad \varphi_2 = \frac{\mathrm{d}z}{z-(x-1)^2}, \qquad \varphi_3 = \frac{\mathrm{d}z}{z-4}$$

Obtain the differential equation system

$$\mathbf{\Omega} = \begin{pmatrix} \frac{d-4}{x-1}+\frac{d-3}{2(x-3)}+\frac{d-3}{2x}+\frac{d-3}{2(x+1)} & \frac{3-d}{2x}+\frac{d-3}{x-1} & \frac{3-d}{2(x-3)}+\frac{3-d}{2(x+1)}+\frac{d-3}{x-1} \\ \frac{3-d}{2x}+\frac{d-3}{x+1} & \frac{d-4}{x+1}+\frac{d-3}{2(x-1)}+\frac{d-3}{2x}+\frac{d-3}{2(x+3)} & \frac{3-d}{2(x-1)}+\frac{3-d}{2(x+3)}+\frac{d-3}{x+1} \\ \frac{3-d}{2(x-3)}+\frac{3-d}{2(x+1)} & \frac{3-d}{2(x-1)}+\frac{3-d}{2(x+3)} & \frac{d-3}{2(x-3)}+\frac{d-3}{2(x-1)}+\frac{d-3}{2(x+1)}+\frac{d-3}{2(x+3)} \end{pmatrix}$$

Its solution contains complete elliptic functions $\mathcal{K}, \mathcal{E}$ consistently with the literature

Quadratic Relations

Define **Identity operators**

$$\mathbb{I}_c = \sum_{i,j} |e_i\rangle \left(\mathbf{C}^{-1}\right)_{ij} \langle e_j| \qquad \mathbb{I}_h = \sum_{i,j} |\mathcal{C}_i] \left(\mathbf{H}^{-1}\right)_{ij} [\mathcal{C}_j| \qquad \mathbf{H}_{ij} = [\mathcal{C}_i|\mathcal{C}_j]$$

Riemann Twisted Period Relation

[Cho, Matsumoto (1995)]

[Mizera (2017)]

$$\langle \varphi_L | \varphi_R \rangle = \sum_{i,j} \langle \varphi_L \overbrace{|\mathcal{C}_i] \left(\mathbf{H}^{-1}\right)_{ij} [\mathcal{C}_j|}^{\mathbb{I}_h} \varphi_R \rangle \quad \Rightarrow \quad \textbf{KLT relation } \mathcal{A}_{closed} \sim \mathcal{A}_{open} \times \mathcal{A}_{open}$$

[Mastrolia, Matsumoto]

$$[\mathcal{C}_L | \mathcal{C}_R] = \sum_{i,j} [\mathcal{C}_L \overbrace{|e_i\rangle \left(\mathbf{C}^{-1}\right)_{ij} \langle e_j|}^{\mathbb{I}_c} \mathcal{C}_R] \quad \Rightarrow \quad \textbf{Elliot's Identity}$$

$$\frac{\Gamma(\lambda+\mu+1)\Gamma(\mu+\nu+1)}{\Gamma(\lambda+\mu+\nu+\frac{3}{2})\Gamma(\mu+\frac{1}{2})} = \begin{pmatrix} F\frac{1}{2}+\lambda, -\frac{1}{2}-\nu, 1+\lambda+\mu; r) \\ F(\frac{1}{2}+\lambda, \frac{1}{2}-\nu, 1+\lambda+\mu; r) \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} F(\frac{1}{2}-\lambda, \frac{1}{2}+\nu, 1+\mu+\nu; 1-r) \\ F(-\frac{1}{2}-\lambda, \frac{1}{2}+\nu, 1+\mu+\nu; 1-r) \end{pmatrix}$$

$\mu = \nu = \lambda = 0$ $\Rightarrow$ **Legendre relation** among Elliptic integrals

$$\frac{\pi}{2} = \begin{pmatrix} \mathcal{E} \\ \mathcal{K} \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \mathcal{K} \\ \mathcal{E} \end{pmatrix} \qquad \mathcal{K}' = \mathcal{K}(r'), \quad \mathcal{E}' = \mathcal{E}(r'), \quad r'^2 + r^2 = 1$$

Summary and Outlook

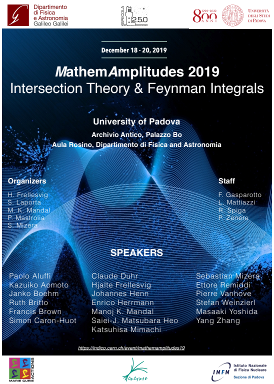
Giving a **new** persective

- Direct decomposition in integral basis and direct construction of system of differential equations
- Algebra of Feynman integrals controlled by intersection number
- Intersection number: Scalar product/Projection between Feynman integrals
- useful for both Physics and Mathematics

A **lot** of new possibilities

- study of **Differential Equations** for Feynman integrals
- application to different representations
- Combine with Finite Fields
- alternatives algorithm for the multivariate intersection number and for the MI reduction
- Quadratic relations ⇔ **Riemann twisted Period Relation**

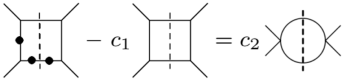
[more at https://indico.cern.ch/event/MathemAmplitudes19]



Thank you for your attention

Top-down approach

[Frellesvig, Gasparotto, Laporta, Mandal, Mastrolia, **L.M.**, Mizera (2020)]



$$\xi = \frac{\sum_{i=-1,j=-1}^{2,2} \kappa_{1ij} x_2^i x_4^j \mathrm{d}x_2 + \sum_{i=-2,j=0}^{2,2} \kappa_{2ij} x_2^i x_4^j \mathrm{d}x_4}{B|_{x_1=x_3=0}}$$

$$\kappa_{1,-1,-1} = \frac{-(d-6)(d-5)t^2}{2s}\,, \qquad \kappa_{1,-1,0} = \frac{(d-6)(d-5)t}{2s}\,,$$

$$\kappa_{1,-1,1} = 0\,, \qquad \kappa_{1,-1,2} = 0\,,$$

$$\kappa_{1,0,-1} = \frac{(3d^2-36d+107)t}{2s}\,, \qquad \kappa_{1,1,-1} = \frac{-(d-7)(3d-17)}{2s}\,,$$

$$\kappa_{1,2,-1} = \frac{(d-7)(d-6)}{2st}\,, \qquad \kappa_{2,-2,0} = \frac{-(d-5)t^2}{2s}\,,$$

$$\kappa_{2,-2,1} = \frac{(d-5)t}{2s}\,, \qquad \kappa_{2,-2,2} = 0\,,$$

$$\kappa_{2,-1,0} = \frac{t(71s-24ds+2d^2s+35t-12dt+d^2t)}{s^2}\,, \qquad \kappa_{2,-1,1} = \frac{-(d-7)(3d-17)}{2s}\,,$$

$$\kappa_{2,-1,2} = \frac{(d-7)(d-6)}{2st}\,, \qquad \kappa_{\text{remain.}} = 0\,.$$

Quadratic relation definitions

$$F(a,b,c;z) = {}_2F_1(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-x\,t)^{-a}dt$$

$\mathcal{K}$ and $\mathcal{E}$ are complete elliptic integrals of the first and second kind

$$\mathcal{K}(r) = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}, 1, r^2\right) = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\phi}{\sqrt{1 - r^2 \sin^2\phi}}$$

$$\mathcal{E}(r) = \frac{\pi}{2} F\left(-\frac{1}{2}, \frac{1}{2}, 1, r^2\right) = \int_0^{\frac{\pi}{2}} \sqrt{1 - r^2 \sin^2\phi}\,\mathrm{d}\phi$$

with $r \in (0,1)$