MMP 2
Solutions 8

FS 19
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Exercise 1

a) A group is a set G together with a map $\cdot : G \times G \mapsto G$ such that:

   i) $a \cdot (b \cdot c) = (a \cdot b) \cdot c \quad \forall a, b, c \in G$

   ii) $\exists e \in G : \quad g \cdot e = e \cdot g = g \quad \forall g \in G$

   iii) $\forall g \in G \quad \exists g^{-1} \in G : \quad g \cdot g^{-1} = g^{-1} \cdot g = 1$

b) A subset H of a group G is a subgroup ($H \leq G$) if it is closed under the multiplication of G, restricted to H, i.e.: $\cdot|_H : H \times H \mapsto G$

c) A subgroup H of G is invariant (under conjugation) or normal, if:
$ghg^{-1} \in H \quad \forall h \in H \quad \forall g \in G$ or equivalently: $gHg^{-1} = H \Leftrightarrow gH = Hg$.

d) Let H be a normal subgroup of G ($H \triangleleft G$). Then the factor group (G modulo H) is defined as the set ${}^{G}/_{H} := \{gH | g \in G\}$ together with the group multiplication of G, generalized to H-cosets:

$$gH \cdot g'H \overset{H \triangleleft G}{=} g\underbrace{g'Hg'^{-1}}_{=H}g'H = gg'H,$$

where the left-cosets are equivalence classes $gH := \{gh | h \in H\} = [g]_\sim := \{g' \in G | g' \sim g\}$ under the equivalence relation $g \sim g' \Leftrightarrow \exists h \in H : \quad g = g'h$.

Exercise 2

$\mathbf{SL}(\mathbf{2}, \mathbb{C})$ is all 2×2 matrices with complex numbers and determinant 1.

Group Properties:

1. Closed under multiplication:
   Let A, $B \in \mathbf{SL}(\mathbf{2}, \mathbb{C})$
   AB is also 2×2 and
   $\det(AB) = \det(A)\det(B) = 1 \cdot 1 = 1$

2. associative multiplication:
   $(A \cdot B) \cdot C = A \cdot (B \cdot C) \qquad A, B, C \in \mathbf{SL}(\mathbf{2}, \mathbb{C})$
   because matrix multiplication is associative

3. $\exists$ unique neutral element:
   $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is 2×2 and $\det(I) = 1 \quad \Rightarrow \; I \in \mathbf{SL}(\mathbf{2}, \mathbb{C})$
   and $IA = AI \qquad \forall A \in \mathbf{SL}(\mathbf{2}, \mathbb{C})$

4. $\exists$ unique inverse $\forall A \in \mathbf{SL}(\mathbf{2}, \mathbb{C})$:
   Every matrix A has a unique inverse A^{-1}. If A is 2×2, A^{-1} must be 2×2 so that $AA^{-1} = A^{-1}A = I$
   and $\det(A^{-1}) = \frac{1}{\det(A)} = 1$
   $\Rightarrow \; A^{-1} \in \mathbf{SL}(\mathbf{2}, \mathbb{C})$

$\mathbf{SL}(\mathbf{2}, \mathbb{C})$ has 4 complex parameters $\qquad \Rightarrow$ 8 real parameters

For $A \in \mathbf{SL}(\mathbf{2}, \mathbb{C})$, $\det(A) = -1$

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\qquad$ a, b, c, d complex
$a = a_x + ia_y, \quad a_x, a_y$ real

$$\begin{aligned} \Rightarrow \quad 1 &= \det(A) \\ &= ac - bd \\ &= (a_x + ia_y)(c_x + ic_y) - (b_x + ib_y)(d_x + Id_y) \\ &= a_x c_x - a_y c_y - b_x d_x + b_y d_y + i(a_y c_x + a_x c_y - b_x d_y - b_y d_x) \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad 1 &= a_x c_x - a_y c_y - b_x d_x + b_y d_y \\ &\text{and} \\ 0 &= a_y c_x + a_x c_y - b_x d_y - b_y d_x \end{aligned}$$

$\Rightarrow$ 2 constants

$\Rightarrow$ $\boxed{\textbf{6 independent real parameters}}$

Exercise 3

Let $\mathcal{R}_z := \{R_z(\phi) | \phi \in \mathbb{R}\}$, with $\quad R_z(\phi) := \begin{pmatrix} cos(\phi) & -sin(\phi) & 0 \\ sin(\phi) & cos(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix}$

be the set of rotations around the z-axis.

1. $\mathcal{R}_z$ is a nonempty subset of SO(3):

$$R_z(0) = \mathbb{1} \quad \Rightarrow \quad \mathbb{1} \in \mathcal{R}_z \text{ and thus } \mathcal{R}_z \neq \emptyset \quad (\mathcal{R}_z \text{ contains the neutral element of SO(3)})$$
$$R_z(\phi) \in \mathbb{R}^{3\times 3}, \quad R_z^T(\phi) = R_z(-\phi) = R_z^{-1} \quad \text{using } -\sin\phi = \sin(-\phi)$$
$$\text{and } \det R_z(\phi) = \cos^2\phi + \sin^2\phi = 1 \quad \forall \phi \in \mathbb{R} \quad \Rightarrow \quad \mathcal{R}_z \subseteq \mathrm{SO}(3)$$

2. closure:

$$\begin{aligned} R_z(\phi)R_z(\Theta) &= \begin{pmatrix} cos(\phi) & sin(\phi) & 0 \\ -sin(\phi) & cos(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} cos(\Theta) & sin(\Theta) & 0 \\ -sin(\Theta) & cos(\Theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos(\phi)\cos(\Theta) - \sin(\phi)\sin(\Theta) & \cos(\phi)\sin(\Theta) + \sin(\phi)\cos(\Theta) & 0 \\ -\sin(\phi)\cos(\Theta) - \cos(\phi)\sin(\Theta) & -\sin(\phi)\sin(\Theta) + \cos(\phi)\cos(\Theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &\overset{(*)}{=} \begin{pmatrix} \cos(\phi+\Theta) & \sin(\phi+\Theta) & 0 \\ -\sin(\phi+\Theta) & \cos(\phi+\Theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= R_z(\phi+\Theta) \in \mathcal{R}_z \end{aligned} \tag{1}$$

$$\Rightarrow \quad \textbf{Multiplication is closed}$$

$$(*) : \begin{pmatrix} \sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta) \\ \cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta) \end{pmatrix}$$

3. The inverse follows from the $A^T A = \mathbb{1}$ property of SO(3) shown above.

$\Rightarrow \quad \mathcal{R}_z \leq \mathrm{SO}(3)$.

Is $\mathcal{R}_z$ abelian?

$$R_z(\phi)R_z(\Theta) = R_z(\phi+\Theta) \overset{(1)}{=} R_z(\Theta+\phi) = R_z(\Theta)R_z(\phi) \quad \Rightarrow \quad R_z \textbf{ is abelian}$$

Is $\mathcal{R}_z$ normal?

$$R_x(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi) & \sin(\phi) \\ 0 & -\sin(\phi) & \cos(\phi) \end{pmatrix} \in \mathbf{SO(3)}$$

Consider rotation by $\frac{\pi}{2}$ around the x-axis:

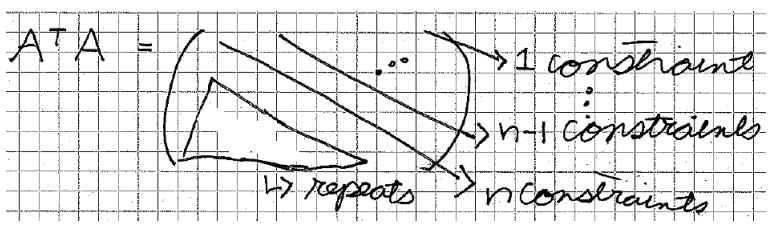
$$\begin{aligned} R_x(\frac{\pi}{2})R_z(\phi)R_x^{-1}(\frac{\pi}{2}) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} cos(\phi) & sin(\phi) & 0 \\ -sin(\phi) & cos(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} cos(\phi) & 0 & -sin(\phi) \\ -sin(\phi) & 0 & -cos(\phi) \\ 0 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} cos(\phi) & 0 & -sin(\phi) \\ 0 & 1 & 0 \\ sin(\phi) & 0 & cos(\phi) \end{pmatrix} \overset{\text{i.g.}}{\notin} \mathcal{R}_z \end{aligned}$$

Exercise 4

a) $\mathbf{O(n)} := \{A \in \mathrm{GL}(n, \mathbb{R}) | A^T A = \mathbb{1}\}$; group of orthogonal $n \times n$ matrices.
$n \times n$ matrices $\quad \Rightarrow \quad n^2$ parameters.

$$\text{orthogonal} \quad \Rightarrow \quad (A^T A)_{jk} = \sum_{l=1}^{n} a_{lj} a_{lk} = \delta_{jk} \quad (a_{jk}\text{elements of } A)$$

The sector above the diagonal of $A^T A$ is the same as below.
Therefore we only need to sum over the upper triangular matrix, including the diagonal:



$$\Rightarrow \#\text{constraints} = n + (n-1) + \ldots + 2 + 1 = \frac{n(n+1)}{2}$$

$$\Rightarrow \textbf{\#independent real parameters} = n^2 - \frac{n(n+1)}{2} = \boxed{\frac{n(n-1)}{2}}$$

b) $\mathbf{U(n)} := \{U \in \mathrm{GL}(n,\mathbb{C}) | U^\dagger U = \mathbb{1}\}$, $U^\dagger = \bar{U}^T$; group of unitary $n \times n$ matrices.
complex $n \times n$ matrices $\quad \Rightarrow \quad 2n^2$ parameters (real + imaginary)

Works almost the same as for $\mathbf{O(n)}$, just that each equation is complex and thus equivalent to two real equations, exept for the diagonal, where the imaginary equations are trivial:

$$\text{unitary} \quad \Rightarrow \quad (U^\dagger U)_{ii} = \sum_j (\bar{u}^i_j)^T u^j_i = \sum_j \bar{u}^j_i u^j_i = \sum_j \underbrace{|u^j_i|^2}_{\in\mathbb{R}} = 1 \tag{1}$$

$$\Rightarrow \quad \text{imaginary equation: } 0i = 0i \quad \Rightarrow \quad \text{no constraint}$$

This does not happen for the off-diagonal entries since i and k are different:

$$i \neq k \quad \Rightarrow \quad (U^\dagger U)_{ik} = (\bar{u}^i_j)^T u^j_k = \underbrace{\bar{u}^j_i u^j_k}_{\substack{i.g. \\ \notin \mathbb{R}}} = 0 \quad \Rightarrow \quad 2 \text{ real constraints} \tag{2}$$

So, we get n real constraints from (1) and we sum to $n-1$ for the upper triangular, to exclude the diagonal:

$$\Rightarrow \quad \#\text{constraints} = n + 2\left[(n-1) + (n-2) + \ldots + 1\right] = n + 2\left[\frac{(n-1)n}{2}\right] = n^2$$

$$\Rightarrow \quad \textbf{\#independent real parameters} = 2n^2 - n^2 = \boxed{n^2}$$

c) $\mathbf{SU(n)} := \{U \in \mathrm{GL}(n,\mathbb{C}) | U^\dagger U = \mathbb{1}, \det U = 1\}$; special group of unitary $n \times n$ matrices
SU(n) < U(n): The determinant gives us one additional real equation, since all eigenvalues λ_i of a unitary matrix are real:

$$\det U = \prod_i \lambda_i = 1 \quad \Rightarrow \quad \textbf{\#independent real parameters} = \boxed{n^2 - 1}$$

An example for an orthogonal matrix which is not in $\mathbf{SO(n)}$ could be:

$$A = \begin{pmatrix} -1 & 0 & & \ldots & 0 \\ 0 & 1 & & 0 & \vdots \\ & & \ddots & & \\ \vdots & 0 & & 1 & 0 \\ 0 & \ldots & & 0 & 1 \end{pmatrix} \qquad \begin{aligned} &\Rightarrow A^T A = \mathbb{1}, && A \in \mathbf{O(n)} \\ &\Rightarrow \det(A) = -1, && A \notin \mathbf{SO(n)} \end{aligned}$$