Elementary Particle Theory – PHY452

Fall Semester 2020

Exercise Sheet 6

Exercise 1: $SU(N)$ **gauge theory**

The non-abelian gauge invariant Lagrangian for a multiplet fermion field $\psi = (\psi_1, \ldots, \psi_N)^T$ reads

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \bar{\psi}(i\not{D} - m)\psi \, , \tag{1}$$

with the field-strength tensor defined as $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - f^{abc} A^b_\mu A^c_\nu$, where A^a_μ are the gauge fields, and the covariant derivative is defined as $(D_\mu)_{ij} = \delta_{ij}\partial_\mu - igT^a_{ij}A^a_\mu$. Here, the matrices T^a are generators of the non-abelian gauge group $SU(N)$ with structure constants f^{abc}.

a) Derive the equations of motion for the fermion fields, expressing all derivatives in terms of the covariant derivative.

b) Derive the equations of motion for the gauge fields.

c) Extract from the generalized Maxwell equations derived in (b), or using Noether's theorem, the conserved current and compare to the abelian case (i.e. QED).

Exercise 2: **More on** $SU(N)$ **gauge theory**

a) Compute the Feynman rules for the three interaction vertices $A\bar{\psi}\psi$, AAA and $AAAA$ for the non-abelian gauge theory in (1).

Hint: For the AAA vertex, assume that each momentum is flowing inward towards the vertex (this is a convention) and perform the replacement $\partial_\mu \to -ik_\mu$ for the interaction derivative, where k is the four-momentum of the gauge boson.

b) Draw all Feynman diagrams contributing to the following processes:

- i) $\psi\bar{\psi} \to \psi\bar{\psi}$ at tree level.
- ii) $\psi\bar{\psi} \to AA$ at tree level.
- iii) $AA \to AA$ at tree level.
- iv) off-shell process $A^* \to \psi\bar{\psi}$ at one-loop level.

Exercise 3: **Dimensional analysis**

The action

$$S = \int \mathrm{d}^D x\, \mathcal{L}(x) \tag{2}$$

in D spacetime dimensions has zero mass dimensions $[S] = 0$.

a) Use (2) to derive the mass dimension of the Lagrangian density $\mathcal{L}$, and of a scalar field ϕ, fermion field ψ and vector field A, as a function of D. Which are their values for D=4?

b) What is the mass dimension $[\Gamma]$ of the decay rate of an unstable particle? What is the mass dimension $[\sigma]$ of the cross-section of a scattering process?

c) In $D = 4$, write all possible Lorentz and gauge invariant interaction terms of the form $\mathcal{L}_{int} = g\, F(\phi, \partial_\mu\phi, A_\mu, \partial_\mu A_\nu, \bar{\psi}\Gamma\psi)$, where ϕ is a complex scalar field, A_μ is an abelian gauge boson, $\Gamma = \{1, \gamma^\mu, \sigma^{\mu\nu}\}$, g is a generic coupling constant with dimension $-2 \leq [g] \leq 0$ and $F(\cdot)$ is a monomial function.

The (free) neutron is an unstable baryon that decays into a proton, an electron and an anti-neutrino $n \to p\, e^- \bar{\nu}_e$. This radioactive process, known as *β-decay*, can be approximately described by the contact interaction Lagrangian

$$\mathcal{L}_{int} \;=\; -\frac{G_F}{\sqrt{2}} (\bar{\psi}_n \gamma^\mu \psi_p)(\bar{\psi}_e \gamma_\mu \psi_\nu) \tag{3}$$

involving four fermion fields. Here G_F is the Fermi constant already encountered in problem sheet 5.

d) **(optional)** Without explicitly computing the decay rate $\Gamma(n \to p\, e^- \bar{\nu}_e)$, use dimensional analysis to estimate that the mean life-time of the free neutron is in the order of a few seconds.

Hints: Use that the amplitude $i\mathcal{M} \propto G_F$ and the coupling has mass dimension $[G_F] < 0$. The estimated decay rate should depend on the mass of the neutron, the mass of the proton and the coupling G_F (you can neglect the masses of the electron and neutrino), which you can look up in the 'N Baryons' section of the [Particle Data Group]. Try to include the factors of 2π coming from the $1 \to 3$ phase-space in your estimation.