Elementary Particle Theory – PHY452

Fall Semester 2020

Exercise Sheet 11

Exercise 1: **Georgi-Glashow $SU(2)$ model**

The Georgi-Glashow model was proposed in 1972 as an alternative to the $SU(2)_L \times U(1)_Y$ to describe the weak interactions. As we will show in this exercise, it was excluded by the discovery of weak neutral currents in electron-neutrino scattering at the Gargamelle experiment (CERN). The gauge group of this model is $SU(2)$ and its Lagrangian is given by

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right) + \mathrm{Tr}\left(D_\mu\phi\, D^\mu\phi\right) + \mu^2\,\mathrm{Tr}\left(\phi^2\right) - \lambda\left[\mathrm{Tr}\left(\phi^2\right)\right]^2 \tag{1}$$

where $\phi = \phi^a\, T^a$ is a real scalar in the adjoint representation and $T^a = \sigma^a/2$ are the $SU(2)$ generators.

a) How does ϕ and its components ϕ^a transform under $SU(2)$? Write down explicitly the covariant derivative $D_\mu\phi^a$.

b) Which values of ϕ minimize the potential?

c) Let us choose the ground state $\langle\phi^1\rangle = \langle\phi^2\rangle = 0$ and $\langle\phi^3\rangle = v$. Find the broken and unbroken generators in this case, and the corresponding symmetry groups.

d) In the unitary gauge, one can use the parameterization $\phi^1 = \phi^2 = 0$ and $\phi^3 = v + h$, where h is a scalar boson that acquires mass via spontaneous symmetry breaking. Show that two gauge bosons get a mass $m_A^2 = g^2 v^2$ and that the third one remains massless.

Exercise 2: **ρ parameter in the SM**

The charged and neutral weak currents in the Standard Model (SM) are defined, respectively, as

$$\begin{aligned} j_\mu^- &= 2\sum_{i=1}^{3} \bar{u}_i^L\gamma_\mu d_i^L + 2\sum_{i=1}^{3}\bar{\nu}_i^L\gamma_\mu \ell_i^L, \qquad j_\mu^+ = \left(j_\mu^-\right)^\dagger\,, \\ j_\mu^Z &= 2\sum_{f=\nu,\ell,u,d}\,\sum_{i=1}^{3}\left[I_f^3\,\bar{f}_i^L\gamma_\mu f_i^L - Q_f\sin^2\theta_W\,\bar{f}_i\gamma_\mu f_i\right]\,, \end{aligned} \tag{2}$$

where fermions are written in the flavor basis, i, j are flavor indices, θ_W is the Weinberg angle and, I_f^3 denotes the third component of weak isospin for a fermion f.

a) Consider the charged current contribution to the scattering process $\nu_i + \ell_j \to \nu_j + \ell_i$ $(i \neq j)$. Starting from the lowest order matrix element, written using the Feynman rules in the full theory, derive the coupling constant of the low-energy effective 4-fermion interaction described by the Lagrangian

$$\mathcal{L}_{\text{eff}}^{\text{CC}} = G^{\text{CC}}\, j_\mu^+ j^{\mu-}\,. \tag{3}$$

b) Considering the process $\nu_i + \bar{\nu}_i \to \nu_j + \bar{\nu}_j$ $(i \neq j)$ do the same as (a) for the effective neutral current interaction

$$\mathcal{L}_{\text{eff}}^{\text{NC}} = G^{\text{NC}}\, j_\mu^Z j^{\mu Z}\,. \tag{4}$$

c) The ρ parameter can be defined as the ratio of the effective couplings for charged and neutral electroweak interactions, i.e. $\rho = G^{\text{CC}}/G^{\text{NC}}$. What is the value of ρ in the SM?

Exercise 3: $WW \to WW$ **scattering and unitarity**

In this exercise we will derive a theoretical bound on the Higgs boson mass by studying the scattering of electroweak gauge bosons at high energies. The amplitude of a generic process can be decomposed in terms of partial-wave coefficients M_ℓ,

$$\mathcal{M}(\theta) = 16\pi \sum_{\ell=1}^{\infty} (2\ell+1) P_\ell(\cos\theta) \mathcal{M}_\ell\,, \tag{5}$$

where θ is the scattering angle and P_ℓ denote Legendre polynomials, which satisfy $P_\ell(1) = 1$ and the orthonormality conditions,

$$\int_{-1}^{1} \mathrm{d}\cos\theta\, P_\ell(\cos\theta) P_{\ell'}(\cos\theta) = \frac{2}{2\ell+1}\delta_{\ell\ell'}\,. \tag{6}$$

For an elastic scattering, the unitarity of the S-matrix implies the optical theorem,

$$\sigma = \frac{1}{s}\,\text{Im}\left[\mathcal{M}(\theta=0)\right]\,. \tag{7}$$

In other words, the total cross-section is related to the imaginary part of the amplitude evaluated in the forward region (i.e., with $\theta = 0$).

a) Compute the total cross-section for Eq. (5) in terms of the M_ℓ coefficients,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2 s}|\mathcal{M}|^2\,. \tag{8}$$

Then use Eq. (5) and (7) to derive the unitarity bounds, for all ℓ values,

$$|\mathcal{M}_\ell| \leq 1\,. \tag{9}$$

b) Draw the Feynman diagrams contributing to $W^+W^- \to W^+W^-$ and $ZZ \to ZZ$ scattering at tree-level in the SM.

c) One can show that the scattering amplitude for longitudinally polarized W-bosons, which we denote by W_L, behaves as follows for $\sqrt{s} \gg m_W$,

$$\mathcal{M}(W_L^+W_L^- \to W_L^+W_L^-) = -2\sqrt{2}G_F m_H^2 \left(\frac{s}{s-m_H^2} + \frac{t}{t-m_H^2}\right)\,. \tag{10}$$

Compute the $\mathcal{M}_0$ coefficient for this amplitude. Then, derive the Lee-Quigg-Thacker unitarity bound by imposing that $|\mathcal{M}_0| \leq 1$,

$$m_H < \sqrt{4\pi v^2} \approx 870 \text{ GeV}\,. \tag{11}$$

Even though the Higgs mass is a free parameter in the SM, this bound was an indication before the LHC that the Higgs-boson H should exist in the TeV range, otherwise the SM would violate unitarity.