MMP II
Exercise Sheet 2

FS 19
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https://www.physik.uzh.ch/en/teaching/PHY322

Issued: 26.2.2019
Due: 5.3.2019

Exercise 1 [Complex numbers (4 points)]
$z, z_1, z_2 \in \mathbb{C}$ are complex numbers.

a) Prove the following identities algebraically:

$$\begin{aligned} |z_1| - |z_2| &\leq |z_1 + z_2| \leq |z_1| + |z_2|, \\ ||z_1| - |z_2|| &\leq |z_1 - z_2|. \end{aligned}$$

b) Find the conditions on z for which the following inequalities hold:

$$|z - 1| < \sqrt{|z^2 - 1|} < |z + 1|.$$

Exercise 2 [Trigonometric and the hyperbolic functions (4 points)]
Assume that the trigonometric functions and the hyperbolic functions are defined for complex arguments by the power series

$$\begin{aligned} \sin z &= \sum_{n=1,\,\text{odd}}^{\infty} (-1)^{(n-1)/2} \frac{z^n}{n!} = \sum_{s=0}^{\infty} (-1)^s \frac{z^{2s+1}}{(2s+1)!}, \\ \cos z &= \sum_{n=0,\,\text{even}}^{\infty} (-1)^{n/2} \frac{z^n}{n!} = \sum_{s=0}^{\infty} (-1)^s \frac{z^{2s}}{(2s)!}, \\ \sinh z &= \sum_{n=1,\,\text{odd}}^{\infty} \frac{z^n}{n!} = \sum_{s=0}^{\infty} \frac{z^{2s+1}}{(2s+1)!}, \\ \cosh z &= \sum_{n=0,\,\text{even}}^{\infty} \frac{z^n}{n!} = \sum_{s=0}^{\infty} \frac{z^{2s}}{(2s)!}, \\ e^z &= \sum_{s=0}^{\infty} \frac{z^s}{s!}. \end{aligned}$$

a) Show that

$$\begin{aligned} i \sin z &= \sinh(iz), \quad \sin(iz) = i \sinh z, \\ \cos z &= \cosh(iz), \quad \cos(iz) = \cosh z. \end{aligned}$$

b) Verify that the familiar functional relations

$$\sinh z = \frac{e^z - e^{-z}}{2} \quad \text{and} \quad \cosh z = \frac{e^z + e^{-z}}{2}$$

still hold in the complex plane.

Exercise 3 [Addition theorems (6 points)]
Using the identities

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

and the analogous ones for $\sinh z$ and $\cosh z$ to show that:

a)

$$\begin{aligned} \sin(x + i\,y) &= \sin x \cosh y + i \cos x \sinh y, \\ \cos(x + i\,y) &= \cos x \cosh y - i \sin x \sinh y, \end{aligned}$$

b)

$$\begin{aligned} |\sin(x + i\,y)|^2 &= \sin^2 x + \sinh^2 y, \\ |\cos(x + i\,y)|^2 &= \cos^2 x + \sinh^2 y. \end{aligned}$$

This demonstrates that we may have $|\sin z| > 1$ and $|\cos z| > 1$ in the complex plane.

Exercise 4 (1 point)
Proof that

$$|\sin z| \geq |\sin x|, \quad |\cos z| \geq |\cos x|,$$

where $z = x + iy$.