Exercise 1. *Axial anomaly in massive QED*

Consider massive QED in 4 dimensions:

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\not{D}\psi - m\bar{\psi}\psi\,. \tag{1}$$

Compute the divergence of the axial current, i.e. $\partial_\mu J^{\mu 5}$*, with*

$$J^{\mu 5} = \bar{\psi}\gamma^\mu\gamma^5\psi\,. \tag{2}$$

Hint: in order to get a non-vanishing result even for vanishing fermion mass, you have to define the axial current as

$$\operatorname*{symm\,lim}_{\delta\to 0}\left(\bar{\psi}\left(x+\frac{\delta}{2}\right)\gamma^\mu\gamma^5\exp\left[-ie\int_{x-\delta/2}^{x+\delta/2}dzA(z)\right]\psi\left(x-\frac{\delta}{2}\right)\right)\,, \tag{3}$$

where the symmetric limit means

$$\operatorname*{symm\,lim}_{\delta\to 0}\left(\frac{\delta^\mu}{\delta^2}\right) = 0, \quad \operatorname*{symm\,lim}_{\delta\to 0}\left(\frac{\delta^\mu\delta^\nu}{\delta^2}\right) = \frac{g^{\mu\nu}}{4}. \tag{4}$$

Solution. We compute the divergence of the axial current by starting from the definition of $J^{\mu 5}$ as the symmetric limit of the composite operator given in eq. (3),

$$\begin{aligned}
\partial_\mu J^{\mu 5} \equiv & \operatorname*{symm\,lim}_{\delta\to 0}\left\{\bar{\psi}\left(x+\delta/2\right)\gamma^\mu\gamma^5\exp\left[-ie\int_{x-\delta/2}^{x+\delta/2}dzA(z)\right]\psi\left(x-\delta/2\right)\right\}\\
= & \operatorname*{symm\,lim}_{\delta\to 0}\left\{\bar{\psi}\left(x+\delta/2\right)\overleftarrow{\not{\partial}}\gamma^5\exp\left[-ie\int_{x-\delta/2}^{x+\delta/2}dzA(z)\right]\psi\left(x-\delta/2\right)\right.\\
& + \bar{\psi}\left(x+\delta/2\right)\gamma^\mu\gamma^5\partial_\mu\left(\exp\left[-ie\int_{x-\delta/2}^{x+\delta/2}dzA(z)\right]\right)\psi\left(x-\delta/2\right)\\
& \left. - \bar{\psi}\left(x+\delta/2\right)\gamma^5\exp\left[-ie\int_{x-\delta/2}^{x+\delta/2}dzA(z)\right]\not{\partial}\psi\left(x-\delta/2\right)\right\}.
\end{aligned} \tag{S.1}$$

In eq. (S.1) we have denoted by $\overleftarrow{\not{\partial}}$ the derivative acting on the function on its left and, in the third summand, we have made use of $\{\gamma^\mu, \gamma^5\} = 0$. The derivatives of the fermionic fields can be obtained from the equation of motion of the QED Lagrangian,

$$\begin{aligned}
\bar{\psi}\left(x+\delta/2\right)\overleftarrow{\not{\partial}} &= i\,\bar{\psi}\left(x+\delta/2\right)\left(e\not{A}\left(x+\delta/2\right)+m\right)\,,\\
\not{\partial}\psi\left(x-\delta/2\right) &= -i\,\left(e\not{A}\left(x-\delta/2\right)+m\right)\psi\left(x-\delta/2\right)\,.
\end{aligned} \tag{S.2}$$

In addition, we can write the derivative of the Wilson line (up to $\mathscr{O}(\delta^2)$ terms) as

$$\begin{aligned}
\partial_\mu\exp\left(-ie\int_{x-\delta/2}^{x+\delta/2}dzA(z)\right) &\simeq \partial_\mu\left(1-ie\int_{x-\delta/2}^{x+\delta/2}dzA(z)\right)\\
&\simeq \partial_\mu\left(1-ieA_\nu(x)\delta^\nu\right) = -ie\partial_\mu A_\nu(x)\delta^\nu\,.
\end{aligned} \tag{S.3}$$

By inserting these results into eq. (S.1), we obtain

$$\partial_\mu J^{\mu 5} = \text{symm}\lim_{\delta\to 0}\left\{\bar\psi\left(x+\delta/2\right)\left[i\left(e\not{A}\left(x+\delta/2\right)-e\not{A}\left(x-\delta/2\right)+2m\right)\exp\left(-ie\int_{x-\delta/2}^{x+\delta/2}dz A(z)\right)\right.\right.$$
$$\left.\left. - ie\not{\partial}A_\nu(x)\delta^\nu\right]\gamma^5\psi\left(x-\delta/2\right)\right\}. \tag{S.4}$$

If we now observe that

$$\not{A}\left(x+\delta/2\right)-\not{A}\left(x-\delta/2\right)\simeq\partial_\nu\not{A}(x)\delta^\nu \tag{S.5}$$

and that, if we work modulo higher order terms in δ, we can replace the Wilson line by 1, we arrive at

$$\begin{aligned}\partial_\mu J^{\mu 5} &= \text{symm}\lim_{\delta\to 0}\left\{\bar\psi\left(x+\delta/2\right)\left[2i\,m+ie\partial_\nu\not{A}(x)\delta^\nu-ie\not{\partial}A_\nu(x)\delta^\nu\right]\gamma^5\psi\left(x-\delta/2\right)\right\}\\ &= \text{symm}\lim_{\delta\to 0}\left\{\bar\psi\left(x+\delta/2\right)\left[2i\,m-ie(\partial_\mu A_\nu(x)-\partial_\nu A_\mu(x))\delta^\nu\gamma^\mu\right]\gamma^5\psi\left(x-\delta/2\right)\right\},\\ &= \text{symm}\lim_{\delta\to 0}\left\{\bar\psi\left(x+\delta/2\right)\left[2i\,m-ieF_{\mu\nu}(x)\delta^\nu\gamma^\mu\right]\gamma^5\psi\left(x-\delta/2\right)\right\}.\end{aligned} \tag{S.6}$$

We observe that the mass-dependent term of eq. (S.6) corresponds to the result for a massive fermion in the classical theory, $\partial_\mu J^{\mu 5} = 2im\,\bar\psi\gamma^5\psi$. This means that the anomalous term depending on the field strength must arise from quantum corrections. Given an arbitrary product of γ-functions Γ, we define the vacuum expectation value

$$\left\langle 0|T\left(\bar\psi(y)\Gamma\psi(z)\right)|0\right\rangle = \text{Tr}\left[-\Gamma\left\langle 0|T\left(\psi(z)\bar\psi(y)\right)|0\right\rangle\right]. \tag{S.7}$$

Hence, in order to evaluate the axial anomaly, we need to compute

$$\begin{aligned}\partial_\mu J_A^{\mu 5} &= \text{symm}\lim_{\delta\to 0}\left\{-ieF_{\mu\nu}(x)\delta^\nu\left\langle 0|T\left(\bar\psi\left(x+\delta/2\right)(-\gamma^\mu\gamma^5)\psi\left(x-\delta/2\right)\right)|0\right\rangle\right\}\\ &= \text{symm}\lim_{\delta\to 0}\left\{ieF_{\mu\nu}(x)\delta^\nu\,\text{Tr}\left[\gamma^\mu\gamma^5\left\langle 0|T\left(\psi\left(x-\delta/2\right)\bar\psi\left(x+\delta/2\right)\right)|0\right\rangle\right]\right\}\end{aligned} \tag{S.8}$$

The vacuum expectation value appearing in eq. (S.8) corresponds to the fermion propagator in position-space which, in the free theory, is given by

$$\left\langle 0|T\left(\psi\left(x-\delta/2\right)\bar\psi\left(x+\delta/2\right)\right)|0\right\rangle = \int\frac{d^4k}{(2\pi)^4}\frac{i(\not{k}+m)}{k^2-m^2}e^{-ik\cdot(-\delta)}. \tag{S.9}$$

In momentum-space, the $\delta\to 0$ limit we are interested is translated into a limit for large values of k, which allow as to neglect mass contribution,

$$\begin{aligned}\left\langle 0|T\left(\psi\left(x-\delta/2\right)\bar\psi\left(x+\delta/2\right)\right)|0\right\rangle &\simeq i\int\frac{d^4k}{(2\pi)^4}\frac{\not{k}}{k^2}e^{ik\cdot\delta} = \not{\partial}\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2}e^{ik\cdot\delta}\\ &= \not{\partial}\left(\frac{i}{4\pi^2}\frac{1}{\delta^2}\right) = \frac{-i}{2\pi^2}\frac{\not{\delta}}{\delta^4}.\end{aligned} \tag{S.10}$$

In the second equality, we have used the differentiation properties of the Fourier transform and in the last one we have used $\frac{\partial}{\partial\delta^\sigma} = 2\delta_\sigma\frac{\partial}{\partial\delta^2}$. We observe that, despite being highly singular in the $\delta\to 0$ limit, $\left\langle 0|T\left(\psi\bar\psi\right)|0\right\rangle$ gives a vanishing contribution to the anomaly (S.8), due to

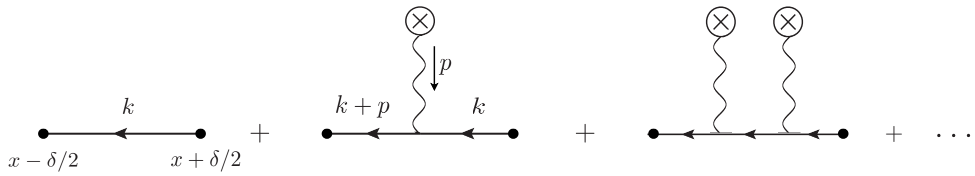

Figure 1: Fermion correlator in presence of a background gauge field.

$\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\gamma^5\right] = 0$. In order to investigate what happens when gauge interactions are switched on, we can study the corrections to the fermion correlator in the presence of a background field $A_\mu(x)$. The corresponding diagrammatic expansion of the propagator is shown in Figure 1. Up to $\mathscr{O}(e^2)$ terms, we have

$$\begin{aligned}\left\langle 0|T\left(\psi\left(x-\delta/2\right)\bar{\psi}\left(x+\delta/2\right)\right)|0\right\rangle_{\text{bkg}} = &\left\langle 0|T\left(\psi\left(x-\delta/2\right)\bar{\psi}\left(x+\delta/2\right)\right)|0\right\rangle \\ &-ie\int d^4x'\left\langle 0|T\left(\psi\left(x-\delta/2\right)\not{A}(x')\bar{\psi}\left(x+\delta/2\right)\right)|0\right\rangle \\ &+\dots \end{aligned} \tag{S.11}$$

The second term of the series corresponds to

$$\begin{aligned}&-ie\int d^4x'\left\langle 0|T\left(\psi\left(x-\delta/2\right)\not{A}(x')\bar{\psi}\left(x+\delta/2\right)\right)|0\right\rangle = \\ &=-ie\int d^4x'\int\frac{d^4k}{(2\pi)^4}\int\frac{d^4p}{(2\pi)^4}e^{-i(k+p)\cdot(x-\delta/2-x')}e^{-ik\cdot(x'-x-\delta/2)}\frac{i(\not{k}+\not{p}+m)\not{A}(x')i(\not{k}+m)}{((k+p)^2-m^2)\left(k^2-m^2\right)} \\ &=ie\int d^4x'e^{ip\cdot x'}\int\frac{d^4k}{(2\pi)^4}e^{ik\cdot\delta}\int\frac{d^4p}{(2\pi)^4}e^{-ip\cdot x}\frac{(\not{k}+\not{p}+m)\not{A}(x')(\not{k}+m)}{((k+p)^2-m^2)\left(k^2-m^2\right)}\,, \end{aligned} \tag{S.12}$$

where, for small δ, we have set $e^{-ip\cdot(x-\delta/2)} \simeq e^{-ip\cdot x}$. Now we observe that the integral over x' corresponds to the Fourier transform of the gauge field $A(x)$,

$$\tilde{\not{A}}(p) = \int d^4x\not{A}(x)e^{-ip\cdot x}\,. \tag{S.13}$$

In addition, as for the leading term, in the large k limit we can neglect the mass dependence and approximate $\left((k+p)^2-m^2\right)\left(k^2-m^2\right) \simeq k^4$. In this way, be obtain

$$\begin{aligned}&-ie\int d^4x'\left\langle 0|T\left(\psi\left(x-\delta/2\right)\not{A}(x')\bar{\psi}\left(x+\delta/2\right)\right)|0\right\rangle = \\ &=ie\int\frac{d^4k}{(2\pi)^4}e^{ik\cdot\delta}\int\frac{d^4p}{(2\pi)^4}e^{-ip\cdot x}\frac{(\not{k}+\not{p})\tilde{\not{A}}(p)\not{k}}{k^4} \\ &=ie\gamma^\alpha\gamma^\beta\gamma^\sigma\int\frac{d^4k}{(2\pi)^4}e^{ik\cdot\delta}\frac{k_\sigma}{k^4}\int\frac{d^4p}{(2\pi)^4}e^{-ip\cdot x}p_\alpha\tilde{A}_\beta(p)\,. \end{aligned} \tag{S.14}$$

In the last equality we have replaced $(k+p)_\alpha \to p_\alpha$ since, once contracted with the trace, the $k_\alpha k_\sigma$ term would give a vanishing contribution. The integrals over k and p are now completely factorised and we immediately recognise

$$\int\frac{d^4p}{(2\pi)^4}e^{-ip\cdot x}p_\alpha\tilde{A}_\beta(p) = i\partial_\alpha A_\beta(x)\,. \tag{S.15}$$

For the second transform, we have

$$\int \frac{d^4k}{(2\pi)^4} e^{ik\cdot\delta} \frac{k_\sigma}{k^4} = \left(i\frac{-\partial}{\partial\delta^\sigma}\right)\left(\frac{-i}{16\pi^2}\log\delta^2\right) = \frac{-1}{8\pi^2}\frac{\delta_\sigma}{\delta^2}\,. \tag{S.16}$$

$$-ie\int d^4x'\,\langle 0|T\left(\psi\left(x-\delta/2\right) \not{A}(x')\bar{\psi}\left(x+\delta/2\right)\right)|0\rangle = \frac{e}{8\pi^2}\gamma^\alpha\gamma^\beta\gamma^\sigma\partial_\alpha A_\beta(x)\frac{\delta_\sigma}{\delta^2}\,. \tag{S.17}$$

We can now insert this result into eq. (S.8) and obtain

$$\begin{aligned}\partial_\mu J_A^{\mu 5} =&\,\text{symm}\lim_{\delta\to 0}\left\{\frac{ie^2}{8\pi^2}F_{\mu\nu}(x)\partial_\alpha A_\beta(x)\frac{\delta^\nu\delta_\sigma}{\delta^2}\,\text{Tr}\left[\gamma^5\gamma^\alpha\gamma^\beta\gamma^\sigma\gamma^\mu\right]\right\}\\ =&\frac{e^2}{4\pi^2}\epsilon^{\alpha\beta\sigma\mu}F_{\mu\nu}(x)F_{\alpha\beta}(x)\,\text{symm}\lim_{\delta\to 0}\frac{\delta^\nu\delta_\sigma}{\delta^2}\\ =&-\frac{e^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}(x)F_{\alpha\beta}(x)\,,\end{aligned} \tag{S.18}$$

where we have made use of $\text{Tr}\left[\gamma^5\gamma^\alpha\gamma^\beta\gamma^\sigma\gamma^\mu\right] = -4i\,\epsilon^{\alpha\beta\sigma\mu}$. In the second equality, due to the presence of $\epsilon^{\alpha\beta\sigma\mu}$, we have substituted the tensor $\partial_\alpha A_\beta$ with its antisymmetric part $1/2F_{\alpha\beta}$. In addition, in the last equality, we have used the symmetric limit given in eq. (4) as well as the antisymmetry property of the Levi-Civita tensor. Hence, in massive QED, the axial current obeys

$$\partial_\mu J^{\mu 5}(x) = 2i\,m\bar{\psi}(x)\gamma^5\psi(x) - \frac{e^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}(x)F_{\alpha\beta}(x)\,. \tag{S.19}$$

Exercise 2. *Fermion number non-conservation*

1. *In massless QED, the divergence of the axial current is given by the Adler-Bell-Jackiw anomaly:*

$$\partial_\mu J^{\mu 5} = -\frac{e^2}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\,. \tag{5}$$

Using this relation, show that:

$$\Delta N_R - \Delta N_L = \frac{e^2}{2\pi^2}\int \mathrm{d}^4x\;\mathbf{E}\cdot\mathbf{B}\,, \tag{6}$$

where N_R and N_L are the numbers of right- and left-handed fermions, **E** *and* **B** *are the electric and magnetic fields respectively.*

2. *By assuming $A_0 = 0$, show that the Hamiltonian for massless fermions is*

$$H = \int \mathrm{d}^3x\left[\psi_R^\dagger\left(-i\sigma\cdot\mathbf{D}\right)\psi_R - \psi_L^\dagger\left(-i\sigma\cdot\mathbf{D}\right)\psi_L\right]\,, \tag{7}$$

where $\mathbf{D} = \nabla - ie\mathbf{A}$.

3. *Consider a background field given by $A^\mu = \left(0,0,Bx^1,A\right)$, where B is constant and A is constant in space and varying adiabatically in time. In order to diagonalise the Hamiltonian of part 2, one has to find a solution of the eigenvalue equation $-i\sigma\cdot\mathbf{D}\psi_R = E\psi_R$, which can be obtained by writing*

$$\psi_R = \begin{pmatrix}\phi_1(x^1)\\ \phi_2(x^1)\end{pmatrix} e^{i\left(k_2x^2+k_3x^3\right)}\,. \tag{8}$$

Derive the differential equations obeyed by ϕ_1 and ϕ_2, show that they are equivalent to the equation of a simple harmonic oscillator and use this observation to find the single-particle spectrum of the Hamiltonian.

4. *Now assume that the system is set up in a cubic box of length L, with periodic boundary conditions. Then the momenta will be quantised as*

$$k_i = \frac{2\pi n_i}{L} \qquad\qquad n_i \in \mathbb{Z}\,. \tag{9}$$

Show that requiring the oscillation centre to stay inside the box leads to the condition

$$k_2 < eBL\,, \tag{10}$$

so that the degeneracy of each energy level is

$$\frac{eL^2B}{2\pi}\,. \tag{11}$$

5. *Finally, suppose to change the background A adiabatically by the amount*

$$\Delta A^1 = \frac{2\pi}{eL}\,. \tag{12}$$

Show that the vacuum loses right-handed fermions. Similarly, compute the eigenvalues for the left-handed fermions and show that under the same change ΔA^1 the vacuum gains the same number of left-handed fermions. Show that these numbers agree with the global non-conservation law of part 1.

Solution.

1. First, let us recall some basic electrodynamics. The electric and magnetic fields are defined, in terms of the scalar potential ϕ and of the vector potential $\mathbf{A}$, by

$$\begin{aligned}\mathbf{E} &= -\frac{\partial}{\partial t}\mathbf{A} - \nabla\phi\,,\\ \mathbf{B} &= \nabla\times\mathbf{A}\,.\end{aligned} \tag{S.20}$$

In addition, given the four-potential $A_\mu = (\phi, -\mathbf{A})$, one can verify that the explicit components of the covariant electromagnetic tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\mu A_\mu$ are given by

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix}\,, \tag{S.21}$$

or, equivalently, $E_i = F_{0i}$, $B_i = -\frac{1}{2}\epsilon^{ijk}F_{jk}$. In our convention, the three- and four-dimensional Levi-Civita tensors are defined by $\epsilon^{0123} = \epsilon^{123} = 1$. With these definitions we can verify that the Adler-Bell-Jackiw anomaly is related to the scalar product between the electric and magnetic field. In fact, we can write

$$\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} = 4\epsilon^{0ijk}F_{0i}F_{jk} = -8E_iB_i = -8\mathbf{E}\cdot\mathbf{B}\,. \tag{S.22}$$

Next, we recall that the time variation of the number of fermions N is given by

$$\frac{dN}{dt} = \int \mathrm{d}^3x\,\partial_0 J^0\,, \tag{S.23}$$

with J^0 being the time component of the vector current $J^\mu = \bar\psi\gamma^\mu\psi$. If we assume the space components $\mathbf{J}$ to vanish sufficiently fast for $|\mathbf{x}| \to 0$, in such a way that $\int \mathrm{d}^3x\,\partial_i\mathbf{J}^i = 0$, we can integrate over time the previous expression and obtain

$$\Delta N \equiv N(t\to\infty) - N(t\to-\infty) = \int \mathrm{d}^4x\partial_0 J^0 = \int \mathrm{d}^4x\partial_\mu J^\mu\,. \tag{S.24}$$

Furthermore, by using the properties of the projector operators $P_{R,L}$, we can rewrite the axial current $J^{\mu 5}$ as the difference between the vector currents of right-handed and left-handed fermions,

$$J^{\mu 5} = \bar{\psi}\gamma^\mu (P_R - P_L)\psi = \bar{\psi}_R \gamma^\mu \psi_R - \bar{\psi}_L \gamma^\mu \psi_L = J_R^\mu - J_L^\mu \,. \tag{S.25}$$

Hence, we have

$$\Delta N_R - \Delta N_L = \int \mathrm{d}^4 x \left(\partial_\mu J_R^\mu - \partial_\mu J_L^\mu \right) = \int \mathrm{d}^4 x \partial_\mu J^{\mu 5} \,. \tag{S.26}$$

Finally, if we express the divergence of the axial current in terms of the Adler-Bell-Jackiw anomaly and we make use of eq. (S.29), we obtain

$$\Delta N_R - \Delta N_L = \frac{e^2}{2\pi^2} \int \mathrm{d}^4 x \, \mathbf{E} \cdot \mathbf{B} \,. \tag{S.27}$$

2. We start from the definition of the Lagrangian density for massless fermions,

$$\mathscr{L} = i\bar{\psi}\not{D}\psi \,, \quad \text{with} \quad D_\mu = \partial_\mu + ieA_\mu \,. \tag{S.28}$$

$\mathscr{L}$ does not depend on $\partial_0 \bar{\psi}$ and, therefore, the Hamiltonian density $\mathscr{H}$ is given by

$$\begin{aligned} \mathscr{H} &= \frac{\partial \mathscr{L}}{\partial(\partial_0 \psi)} \partial_0 \psi - \mathscr{L} \\ &= i\bar{\psi}\gamma^0 \partial_0 \psi - i\bar{\psi}(\not{\partial} + ie\not{A})\psi \\ &= i\bar{\psi}(\gamma^k \partial_k + ie\gamma^k A_k)\psi + ie\bar{\psi}\gamma^0 A_0 \psi \\ &= i\bar{\psi}(\gamma^k \partial_k - ie\gamma^k \mathbf{A}_k)\psi \,, \end{aligned} \tag{S.29}$$

where the latin index k denotes the space components of the four-vectors. In the last equality, we have used the assumption $A_0 = 0$ as well as the fact that, for the space-components of A_μ, we have $A_k = -\mathbf{A}_k$.

By decomposing $\psi = (\psi_R \,, \psi_L)^T$, we can use the chiral basis of γ-matrices in order to rewrite the bilinear $\bar{\psi}\gamma^k \psi$ as

$$\begin{aligned} \bar{\psi}\gamma^k \psi = \bar{\psi}^\dagger \gamma^0 \gamma^k \psi &= (\psi_R^\dagger \,, \psi_L^\dagger) \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix} \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} \\ &= \psi_R^\dagger (-\sigma^k) \psi_R + \psi_L^\dagger \sigma^k \psi_L \,. \end{aligned} \tag{S.30}$$

Hence, by inserting (S.37) into eq. (S.36), we obtain

$$\mathscr{H} = \psi_R^\dagger (-i\sigma^k)(\partial_k - ie\mathbf{A}_k)\psi_R + \psi_L^\dagger (i\sigma^k)(\partial_k - ie\mathbf{A}_k)\psi_L \,, \tag{S.31}$$

so that, by integrating over space, we arrive at the Hamiltonian

$$H = \int \mathrm{d}^3 x \left[\psi_R^\dagger \left(-i\sigma \cdot \mathbf{D} \right) \psi_R - \psi_L^\dagger \left(-i\sigma \cdot \mathbf{D} \right) \psi_L \right] \,, \quad \text{with} \quad \mathbf{D} = \nabla - ie\mathbf{A} \,.$$

3. We need to solve the Eigenvalue problem $-i\sigma \cdot \mathbf{D}\psi_R = E\psi_R$ which, under the assumption $\mathbf{A} = (0, Bx_1, A)$, becomes

$$\left[-i\sigma_1 \partial_1 - i\sigma_2(\partial_2 - ieBx^1) - i\sigma_3(\partial_3 - ieA) - E \right] \psi_R = 0 \,. \tag{S.32}$$

According to the Ansatz on the exercise sheet, the space derivatives of ψ_R are given by

$$\partial_1\psi_R = \begin{pmatrix}\phi_1'\\ \phi_2'\end{pmatrix} e^{i\left(k_2x^2+k_3x^3\right)}\,, \quad \partial_2\psi_R = ik_2\psi_R\,, \quad \partial_3\psi_R = ik_3\psi_R\,, \tag{S.33}$$

where, for ease of notation, we have denoted $\partial_1\phi_i(x_1) = \phi_i'$.

By inserting these derivatives into eq. (S.39), we obtain a system of coupled first order differential equations for the fields $\phi_{1,2}$,

$$\sigma_1\begin{pmatrix}\phi_1'\\ \phi_2'\end{pmatrix} = -i\left[\sigma_2(k_2-eBx^1)+\sigma_3(k_3-eA)-E\right]\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}\,. \tag{S.34}$$

With the explicit expression of the Pauli matrices, the previous equations can be written in the matrix form,

$$\begin{pmatrix}\phi_2'\\ \phi_1'\end{pmatrix} = \begin{pmatrix}-i(k_3-eA-E) & -(k_2-eBx_1)\\ (k_2-eBx_1) & i(k_3-eA+E)\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}\,. \tag{S.35}$$

Such system is equivalent to a second order homogenous differential equation for one of the two fields ϕ_i. For instance, we can take the second of eq.s (S.42),

$$\phi_1' = (k_2-eBx_1)\phi_1 + i(k3-eA+E)\phi_2\,, \tag{S.36}$$

and use it to express ϕ_2 in terms of ϕ_1 and its first derivative,

$$\phi_2 = \frac{-i}{(k2-eA+E)}\left(\phi_1' - (k_2-eBx_1)\phi_1\right)\,. \tag{S.37}$$

This expression, once inserted in the first of eq.s (S.42), gives ϕ_2' in terms of ϕ_1 and ϕ_1',

$$\begin{aligned}\phi_2' &= -\,i(k_2-eA-E)\phi_1) - (k_2-eBx_1)\phi_2\\ &= \frac{i(k_2-eBx_1)}{(k_3-eA+E)}\phi_1' - \frac{i}{(k_3-eA+E)}\left[(k_2-eBx_1)^2 + \left((k_3-eA)^2-E^2\right)\right]\phi_1\,.\end{aligned} \tag{S.38}$$

On the other hand, we can differentiate both sides of eq. (S.43) and obtain

$$\phi_1'' = (k_2-eBx_1)\phi_1' - eB\phi_1 + i(k3-eA+E)\phi_2'\,. \tag{S.39}$$

If we insert eq. (S.45) in the last expression we arrive at

$$-\,\phi_1'' + (k_2-eBx_1)^2\phi_1 = \left(E^2+eB-(k_3-eA)^2\right)\phi_1, \tag{S.40}$$

which is a second order differential equation for ϕ_1 that doesn't contain any term proportional to ϕ_1' and, hence, it corresponds to the differential equation of a one-dimensional harmonic oscillator centred in $x_0 = k_2/(eB)$. In fact, if we introduce the change of variables

$$y_1 = \frac{1}{\sqrt{eB}}(k_2-eBx_1) \quad\Rightarrow\quad \frac{d^2}{dx_1^2} = eB\frac{d^2}{dy_1^2}\,, \tag{S.41}$$

which leads to

$$-\,\phi_1'' + y_1^2\phi_1 = \frac{\left(E^2+eB-(k_3-eA)^2\right)}{eB}\phi_1\,, \tag{S.42}$$

we can recognise as the Eigenvalue equation for the harmonic oscillator,

$$\left(-\frac{1}{2}\frac{d^2}{dy_1^2}+\frac{1}{2}y_1^2\right)\varphi = \mathscr{E}\varphi, \tag{S.43}$$

with energy $\mathscr{E} = \left(E^2 + eB - (k_3 - eA)^2\right)/(2eB)$. In order to have a square integrable solution we must impose the quantisation condition

$$\mathscr{E}_n = n + \frac{1}{2}, \quad n \in \mathbb{N}_0 , \tag{S.44}$$

which, finally, gives us the one-particle spectrum of the Hamiltonian

$$E_n^2 = (k_3 - eA)^2 + (2neB) , \quad n \in \mathbb{N}_0 . \tag{S.45}$$

4. By requiring that the centre of oscillation $x_0 = k_2/(eB)$ lies within the cubic box of length L, we obtain a bound on the component of the momenta along the x_2 axis,

$$0 < \frac{k_2}{eB} < L \Leftrightarrow 0 < k_2 < eBL. \tag{S.46}$$

Momentum quantisation $k_i = (2\pi n_i)/L$ translates this constraint into a bound on the quantum number n_2,

$$0 < n_2 < \frac{L^2 eB}{2\pi}. \tag{S.47}$$

In addition, if we insert the quantisation law for the momentum components into eq. (S.52),

$$E^2_{(n,n_3)} = \left(\frac{2\pi n_3}{L} - eA\right)^2 + (2neB) , \quad n\, , n_3 \in \mathbb{N}_0 , \tag{S.48}$$

we observe that the spectrum (S.55) is independent of n_2. Hence, according to eq. (S.54), we conclude that each energy level has a $L^2eB/(2\pi)$-fold degeneracy.

5. For simplicity, let us set $n = 0$. In this case, the one-particle spectrum of eq. (S.54) reads

$$E_{(n_3,\pm)} = \pm\left(\frac{2\pi n_3}{L} - eA\right) . \tag{S.49}$$

When we apply an adiabatic shift of the vector potential, $A \to A + \Delta A^1$, the energy level (S.56) changes as

$$E'_{(n_3,\pm)} = \pm\left(\frac{2\pi n_3}{L} - eA - e\Delta A^1\right) = \pm\left(\frac{2\pi(n_3 - 1)}{L} - eA\right) . \tag{S.50}$$

If we imagine to fill all the negative levels and we interpret a hole created within these filled levels as antiparticle states, we see that the transformation of the external potential leaves the energy spectrum unchanged by shifting each of the equispaced energy levels down by one position. Therefore, given that each energy level is $L^2eB/(2\pi)$-times degenerate, $L^2eB/(2\pi)$ right-handed fermions have disappeared from the vacuum after the adiabatic transformation,

$$\Delta N_R = -\frac{eBL^2}{2\pi} . \tag{S.51}$$

With a calculation analogous to the one of point 3, one can solve the eigenvalue problem $i\sigma \cdot \mathbf{D}\psi_L = E\psi_L$ by writing ψ_L as

$$\psi_L = \begin{pmatrix} \phi_1(x^1) \\ \phi_2(x^1) \end{pmatrix} e^{-i\left(k_2x^2 + k_3x^3\right)} . \tag{S.52}$$

It is easy to verify that, in the determination of the energy-spectrum, this amounts to an effective change $\mathbf{A} \to -\mathbf{A}$ into eq. (S.55)

$$E^2_{(n,n_3)} = \left(\frac{2\pi n_3}{L} + eA\right)^2 - (2neB)\,, \quad n\,, n_3 \in \mathbb{N}_0\,, \tag{S.53}$$

Hence, by following the above reasoning, we conclude that, after the adiabatic transformation, the vacuum gains

$$\Delta N_L = \frac{eBL^2}{2\pi} \tag{S.54}$$

left-handed fermions. We can check this result against the Adler-Bell-Jackiw non-conservation law. If we suppose the adiabatic change of the background potential to occur during the time interval $[0, T]$,

$$A(t) = A + \frac{2\pi}{eL}\,\frac{t}{T}\,, \tag{S.55}$$

we have, according to (S.29),

$$\mathbf{E} = \left(0, 0, -\frac{2\pi}{eLT}\right)\,, \quad \mathbf{B} = (0, 0, B)\ . \tag{S.56}$$

Hence, the global variation of the number of right-handed and left-handed fermions (S.34) is

$$\Delta N_R - \Delta N_L = \frac{e^2}{2\pi^2}\int_0^L \mathrm{d}^3\mathbf{x}\int_0^T \mathrm{d}t\,\mathbf{E}\cdot\mathbf{B} = \frac{-e^2}{2\pi^2}\int_0^L \mathrm{d}^3\mathbf{x}\int_0^T \mathrm{d}t\,\left(\frac{2\pi B}{eLT}\right) = -\frac{eBL^2}{\pi}\,, \tag{S.57}$$

which is consistent with eq.s (S.58)-(S.61).