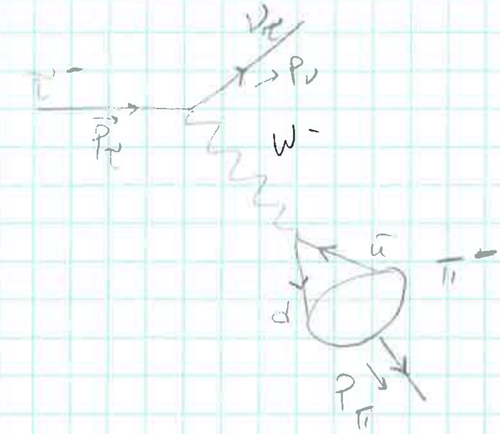
KT II - EXERCISE SHEET 5

TAU DECAY

a) $\tau^- \to \pi^- \nu_\tau$



$\pi^- (\bar{u} d)$

$$-i M_{fi} \approx \left[\bar{u}(p_\nu) \frac{i g_w}{\sqrt{2}} \frac{1}{2} \gamma^\mu (1-\gamma^5) u(p_\tau) \right] \times \frac{i g_{\mu\nu}}{m_W^2} \times \left[\frac{i g_w}{\sqrt{2}} \frac{1}{2} f_\pi p_\pi^\nu \right]$$

$$M_{fi} \approx \left[\bar{u}(p_\nu) \frac{g_w}{\sqrt{2}} \frac{1}{2} \gamma^\mu (1-\gamma^5) u(p_\tau) \right] \cdot \frac{g_{\mu\nu}}{m_W^2} \times \left[\frac{g_w}{\sqrt{2}} \frac{1}{2} f_\pi p_\pi^\nu \right]$$

$$\approx \frac{g_w^2}{8 m_W^2} \left[\bar{u}(p_\nu) \frac{1}{2} \gamma^\mu (1-\gamma^5) u(p_\tau) \right] g_{\mu\nu} \left[f_\pi p_\pi^\nu \right]$$

$$\approx \sqrt{2}\, G_F f_\pi \, \bar{u}(p_\nu) \gamma^\mu \frac{1}{2} (1-\gamma^5) u(p_\tau) g_{\mu\nu} p_\pi^\nu$$

b) $p_\nu = p^* (1, \sin\theta, 0, \cos\theta)$

The neutrino must be produced in a LH helicity state, because the weak interaction selects LH particle states

For the neutrino helicity and chirality states coincide, because they are massless.

The spin of tau particles is along z direction, so we can consider the "spin up" solution of the Dirac equation for this particle in its rest frame.

$\Rightarrow$

$$j^i = \frac{1}{2} 2 \sqrt{2m_\tau p^*} \left[\sigma_{11}^i s - \sigma_{12}^i c \right] \to j^1 = \sqrt{2m_\tau p^*} (-c) \quad \text{because } \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$j^2 = \sqrt{2m_\tau p^*} (-ic) \quad \text{with } \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$j^3 = \sqrt{2m_\tau p^*} (s) \quad \text{with } \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\Rightarrow j^\mu = \sqrt{2m_\tau p^*} (-s, -c, -ic, s)$$

c) $P_\pi = (E, -p^* \operatorname{sen}\theta, 0, -p^* \cos\theta)$

But $\operatorname{sen}\theta = 2 \operatorname{sen}\frac{\theta}{2} \cos\frac{\theta}{2} = 2sc$

$\cos\theta = \cos^2\frac{\theta}{2} - \operatorname{sen}^2\frac{\theta}{2} = c^2 - s^2$

$$\begin{aligned}
j^\mu \cdot p^\pi &= \sqrt{2m_\tau p^*} (-s, -c, -ic, s)(E, -p^* 2sc, 0, -p^*(c^2 - s^2)) \\
&= \sqrt{2m_\tau p^*} (-Es - 2sc^2 p^* + p^* s(c^2 - s^2)) \\
&= \sqrt{2m_\tau p^*} (-Es - 2sc^2 p^* + p^* sc^2 - p^* s^3) \\
&= \sqrt{2m_\tau p^*} (-Es - p^* s(c^2 + s^2)) = \sqrt{2m_\tau p^*}\, s(-E - p^*) = \sqrt{2m_\tau p^*}\, s(-m_\tau)
\end{aligned}$$

So $|M|^2 = 4G_F^2 f_\pi^2 m_\tau^3 p^* \operatorname{sen}^2\frac{\theta}{2}$

d) The expression of decay rate for a 2-body decay is given by:

$$\Gamma = \frac{p^*}{32\pi^2 m_\tau^2} \int |M_{fi}|^2 d\Omega$$

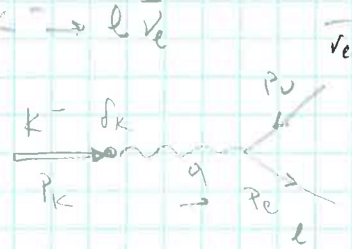
When $\int |M_{fi}|^2 d\Omega = 4G_F^2 f_\pi^2 m_\tau^2 p^* \int_0^{2\pi} \int_{-1}^{1} \left(\frac{1 - \cos\theta}{2} \right) d\cos\theta \, d\varphi =$

$$= 4 G_F^2 f_\pi^2 m_\tau^2 p^* \frac{1}{\cancel{4}} 2\pi \cdot 2 = 8 G_F^2 f_\pi^2 m_\tau^2 p^* \pi$$

$$\Rightarrow \Gamma = \frac{\cancel{8} G_F^2 f_\pi^2 m_\tau^3 \pi}{\cancel{32}_4 \pi^2 m_\tau^2} \cdot (p^*)^2 = \frac{G_F^2 f_\pi^2 m_\tau}{16\pi} \cdot \left(\frac{m_\tau^2 - m_\pi^2}{m_\tau} \right)^2$$

KAON DECAY

$K^- \to \ell \bar{\nu}_\ell$



Kaon rest frame

$$p_K = (m_K, 0, 0, 0) \quad , \quad p_\ell = (E_\ell, 0, 0, p) \quad , \quad p_\nu = (p, 0, 0, -p)$$

$$M_{fi} = \left[\frac{g_W}{\sqrt{2}} \frac{1}{2} f_K p_K^\mu\right] \times \left[\frac{g_{\mu\nu}}{m_W^2}\right] \times \left[\frac{g_W}{\sqrt{2}} \bar{u}(p_\ell)\gamma^\nu \frac{1}{2}(1-\gamma^5) v(p_\nu)\right]$$

$$= \frac{g_W^2}{4 m_W^2} g_{\mu\nu} f_K p_K^\mu \bar{u}(p_\ell)\gamma^\nu \frac{1}{2}(1-\gamma^5) v(p_\nu)$$

$\Rightarrow$ we have approximated the interaction to contact interaction $q^2 = m_K^2 \ll m_W^2$

Given that the only non-zero component of the kaon momentum is the time one $\Rightarrow j_\ell^\nu \Rightarrow j_\ell^0$

$$M_{fi} = \frac{g_W^2}{4 m_W^2} f_K m_K \bar{u}(p_\ell)\gamma^0 \frac{1}{2}(1-\gamma^5) v(p_\nu)$$

But $\bar{u} = u^+ \gamma^0$ and $\gamma^0\gamma^0 = 1$

$$\Rightarrow M_{fi} = \frac{g_W^2}{4 m_W^2} f_K m_K u^+(p_\ell) \frac{1}{2}(1-\gamma^5) v(p_\nu)$$

Neutrinos are massless, so helicity state correspond to chirality states and $\frac{1}{2}(1-\gamma^5) v(p_\nu) = v_\uparrow(p_\nu)$ $\to$ $\frac{1}{2}(1-\gamma^5)$ selects the RH antiparticle state

For the lepton we have both LH and RH helicity state (given it has mass $\neq 0$)

It is easy to verify that the only possible combination is with RH lepton.

$$u_\uparrow(p_\ell) = \sqrt{E_\ell + m_\ell}\begin{pmatrix} 1 \\ 0 \\ \frac{p}{E_\ell + m_\ell} \\ 0 \end{pmatrix} \quad \text{and} \quad u_\downarrow(p_\ell) = \sqrt{E_\ell + m_\ell}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -\frac{p}{E_\ell + m_\ell} \end{pmatrix}$$

While $\quad \bar{v}_\uparrow(p_\nu) = \sqrt{p_\nu}\begin{pmatrix}1\\0\\-1\\0\end{pmatrix}$

$$M_{fi} = \frac{g_w^2}{4m_w^2} f_K m_K \sqrt{E_e+m_e}\,\sqrt{p_\nu}\left(1-\frac{p_\nu}{E_e+m_e}\right)$$

But $\quad m_K = E_e + p_\nu \Rightarrow E_e = m_K - p_\nu$

$$E_e^2 = m_e^2 + p_e^2$$

$$\Rightarrow (m_K - p_\nu)^2 = m_e^2 + p_e^2 \qquad \to \vec{p}_e = -\vec{p}_\nu$$

$$m_K^2 + \cancel{p_\nu^2} - 2m_K p_\nu = m_e^2 + p_\nu^2$$

$$p_\nu = \frac{m_K^2 - m_e^2}{2m_K}$$

$$E_e = \frac{m_K^2 + m_e^2}{2m_K}$$

So $\quad M_{fi} = \frac{g_w^2}{4m_w^2} f_K m_K \frac{m_K+m_e}{\sqrt{2m_K}} \cdot \left(\frac{m_K^2-m_e^2}{2m_K}\right)^{\frac{1}{2}} \cdot \frac{2m_e}{m_K+m_e}$

$$= \left(\frac{g_w^2}{2m_w}\right)^2 f_K m_e (m_K^2 - m_e^2)^{\frac{1}{2}}$$

Kaon is a spin 0 particle $\Rightarrow$ no need to average over initial-state spins

$$\langle|M_{fi}|^2\rangle = |M_{fi}|^2 = 2G_F^2 f_K^2 m_e^2 (m_K^2 - m_e^2)$$

$|M_{fi}|^2$ does not depend on theta $\to$ isotropic matrix element

$$\Gamma = \frac{4\pi}{32\pi^2 m_K^2} p_\nu \langle|M_{fi}|^2\rangle = \frac{G_F^2}{8\pi m_K^3} \cdot f_K^2 \left[m_e (m_K^2 - m_e^2)\right]^2$$

therefore $\quad \frac{\Gamma(K^- \to e^-\bar{\nu}_e)}{\Gamma(K^- \to \mu^-\bar{\nu}_\mu)} = \frac{m_e(m_K^2 - m_e^2)}{m_\mu(m_K^2 - m_\mu^2)}$

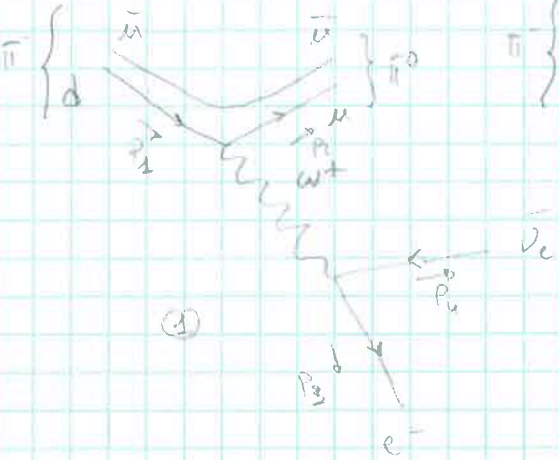
WEAK INTERACTIONS

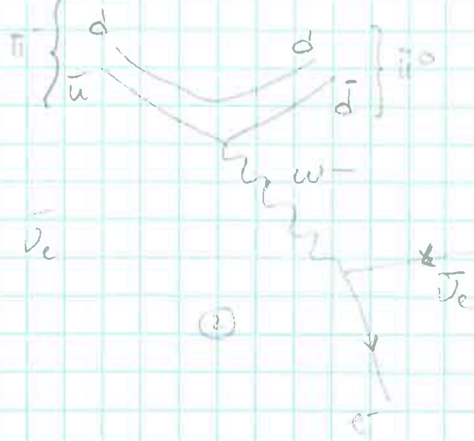
a) $\pi^- \to \pi^0 e^- \bar{\nu}_e$

$d\bar{u} \to u\bar{u}\, e^- \bar{\nu}_e$

or

$d\bar{u} \to d\bar{d}\, e^- \bar{\nu}_e$





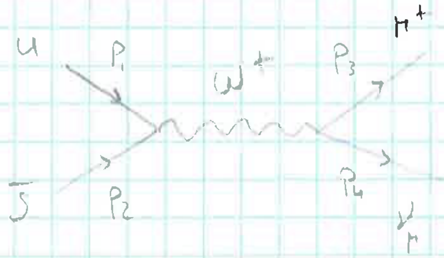
Without considering spectator quarks, so in the limit of deep inelastic scattering the matrix element is

(1) $$-iM_{fi} = \bar{u}(2)\, \frac{i g_W}{\sqrt{2}} \gamma^\mu \frac{1}{2}(1-\gamma^5)\, u(1)\; \frac{i g_{\mu\nu}}{m_W^2}\; \bar{u}(3)\, \frac{i g_W}{\sqrt{2}} \gamma^\nu \frac{1}{2}(1-\gamma^5)\, v(4)$$

$$M_{fi} = \frac{g_W^2}{2 m_W^2} g_{\mu\nu} \left[\bar{u}(2) \frac{1}{2}(1-\gamma^5)\, u(1)\right] \left[\bar{u}(3) \frac{1}{2}(1-\gamma^5)\, v(4)\right]$$

b) $K^+ \to \mu^+ \nu_\mu$

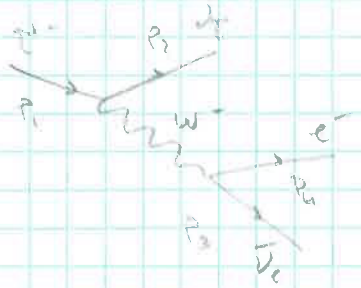
$u\bar{s} \to \mu^+ \nu_\mu$



$$-iM_{fi} = \bar{v}(2)\, \frac{i g_W}{\sqrt{2}} \gamma^\mu \frac{1}{2}(1-\gamma^5)\, u(1)\; \frac{i g_{\mu\nu}}{m_W^2}\; \bar{u}(4)\, \frac{i g_W}{\sqrt{2}} \gamma^\nu \frac{1}{2}(1-\gamma^5)\, v(3)$$

$$M_{fi} = \frac{g_W^2}{2 m_W^2} g_{\mu\nu} \left[\bar{v}(2) \gamma^\mu \frac{1}{2}(1-\gamma^5)\, u(1)\right] \left[\bar{u}(4) \gamma^\nu \frac{1}{2}(1-\gamma^5)\, v(3)\right]$$

c) $\tau^- \to e^- \bar{\nu}_e \nu_\tau$



$$iM_{fi} = \bar{u}(2)\, \frac{i g_W}{\sqrt{2}} \gamma^\mu \frac{1}{2}(1-\gamma^5)\, u(1)\; \frac{i g_{\mu\nu}}{m_W^2}\; \bar{u}(4)\, \frac{i g_W}{\sqrt{2}} \gamma^\nu \frac{1}{2}(1-\gamma^5)\, v(3)$$

$$M_{fi} = \frac{g_W^2 g_{\mu\nu}}{2 m_W^2} \left[\bar{u}(2) \gamma^\mu \frac{1}{2}(1-\gamma^5)\, u(1)\right] \left[\bar{u}(4) \gamma^\nu \frac{1}{2}(1-\gamma^5)\, v(3)\right]$$