MMP II
Exercise Sheet 5

FS 19
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Exercise 1 [Path integral (2 points)]
Calculate

$$\int_\gamma Re(z^n)\mathrm{d}z$$

where $n \in \mathbb{Z}$ and γ is the unit circle in positive orientation. Which n are special?

Exercise 2 [Taylor expansion (4 points)]
Determine the Taylor expansion for

$$\frac{1}{\sqrt[5]{z+i}}$$

around $z = a$ for $a \neq -i$ and find the convergence radius. Assume the convention that

$$\sqrt[5]{z} \equiv \sqrt[5]{r\,\exp(i\,\phi)} = \sqrt[5]{r}\,\exp(i\,\phi/5) \quad -\pi < \phi \leq \pi,$$

i.e. we only work with the zero-order root of $\sqrt[5]{z}$. This makes the above Taylor expansion well-defined and analytic.

Hint: The convergence radius can be computed with Hadamard's formula:

$$\rho = \lim_{k\to\infty}\left|\frac{c_k}{c_{k+1}}\right|,$$

where the c_k are the coefficients of the Taylor series.

– please turn over –

Exercise 3 [Laurent series (6 points)]
Show that

$$f(z) = \frac{1}{z^4 - 1}$$

can be expanded around $z = 1$ in the Laurent series

$$\frac{1}{4}(z-1)^{-1} + \frac{1}{4}\sum_{k=0}^{\infty}\left[\frac{-i(i+1)^{k+1} - i(-1)^k(i-1)^{k+1} + 1}{(-2)^{k+1}}\right](z-1)^k$$

and determine the radius of convergence.

Hint: Decompose $f(z)$ into partial fractions of its roots and try to find Laurent series for each summand by putting it into an appropriate formula such that it can be described using a geometric series:

$$\frac{1}{1-r} = \sum_{k=0}^{\infty} r^k \qquad r \in \mathbb{C},\ |r| < 1$$

Exercise 4 [Cauchy Principal Value (7 points)]
Show that the limit (principal value)

$$\lim_{\varepsilon \to 0}\left(\int_0^{1-\varepsilon} + \int_{1+\varepsilon}^{\infty}\right)\frac{\cos(a\,x)}{x^2-1}\mathrm{d}x$$

for $a \in \mathbb{R}^+$ exists and determine its value.

Hint: Write the above integral as

$$I = \lim_{\varepsilon \to 0}\left(\int_{-\infty}^{-1-\varepsilon} + \int_{-1+\varepsilon}^{0} + \int_0^{1-\varepsilon} + \int_{1+\varepsilon}^{\infty}\right)\frac{e^{iax}}{2(x^2-1)}\,dx$$

Instead of x, use a complex argument z (analytic continuation) and apply Cauchy's Theorem, constructing a closed curve going around the poles ± 1 with infinitesimal circles $\gamma_{\varepsilon,\delta}$ in the positive half-plane. This will give you the integral I.