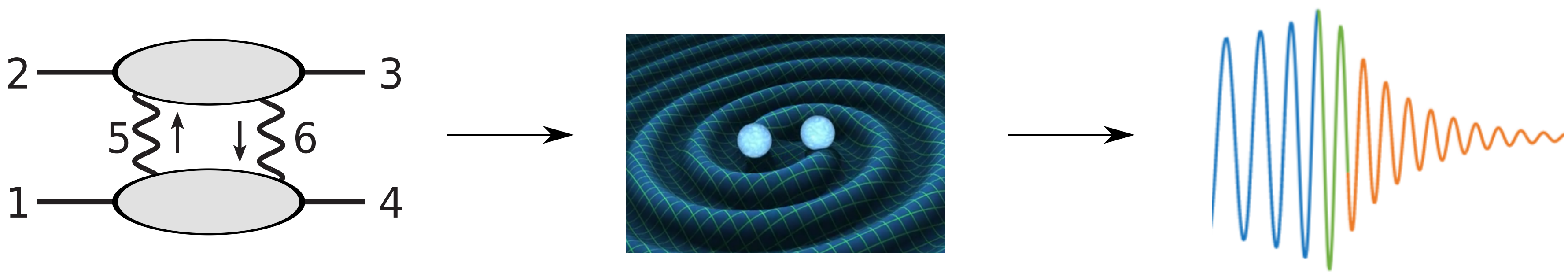
ETH zürich

Black Hole Binary Dynamics from Scattering Amplitudes



Mao Zeng, Institute for Theoretical Physics, ETH Zürich

Seminar at University of Zürich, 03 Dec 2019

arXiv: 1901.04424 (PRL), arXiv:1908.01493 (JHEP),
Zvi Bern, Clifford Cheung, Radu Roiban, Chia-Hsien Shen, Mikhail P. Solon, MZ

Work in progress, Harald Ita, Michael Ruf, MZ

OUTLINE

1. Introduction

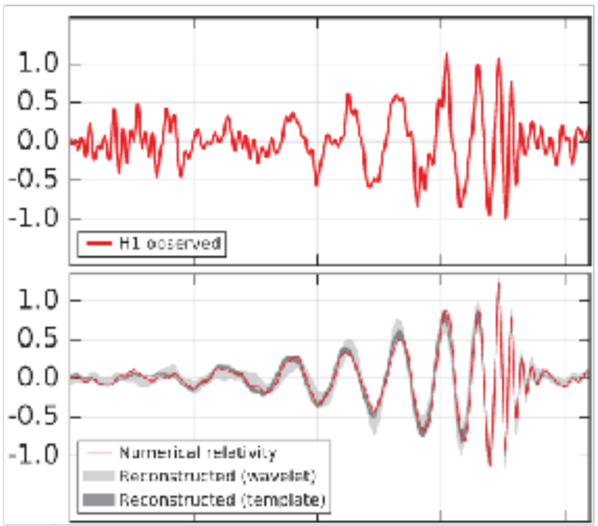
2. Scattering amplitudes - double copy & unitarity cuts

3. Potential from non-relativistic EFT

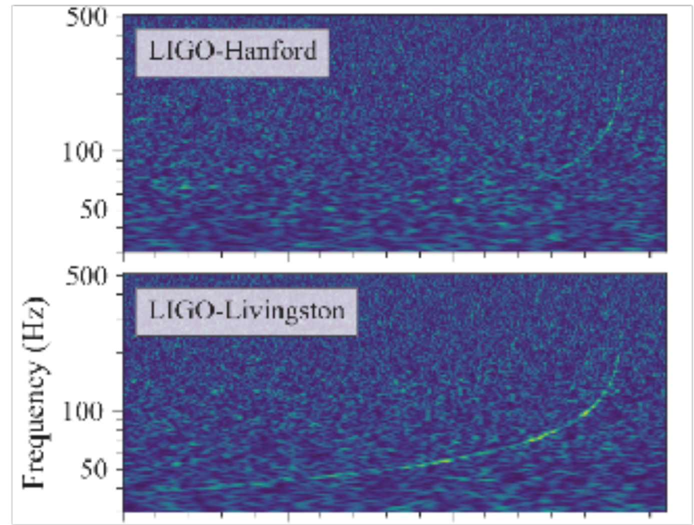
4. Relativistic integration - importing techniques from precision QCD

BIRTH OF AN ERA

- LIGO / VIRGO detected gravitational waves: BH-BH (2015), BH-NS (2017), NS-NS (2019?)



LIGO & VIRGO, arXiv:1602.03837



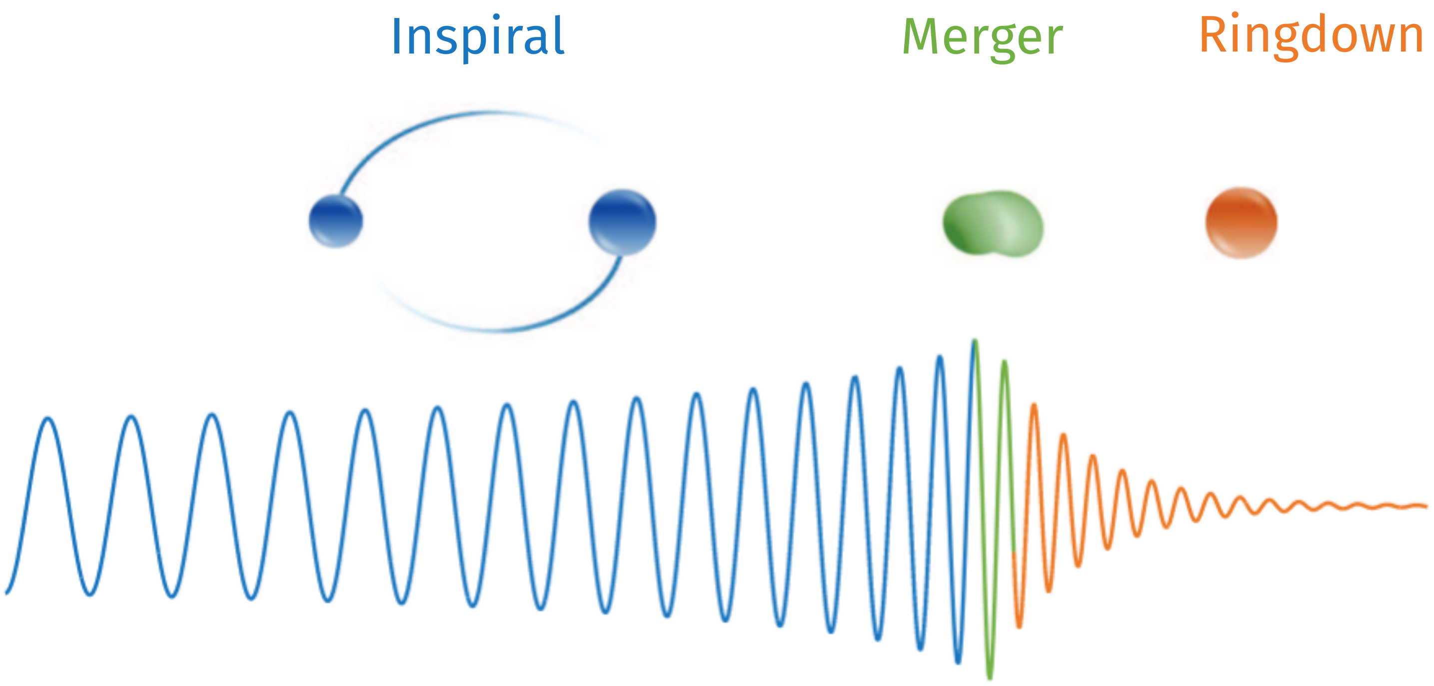
LIGO & VIRGO, arXiv:1710.05832

- Next-gen. experiments (LISA, CE, ET...): high S-N ratio, dominated by *theory uncertainty*.

- precision predictions necessary

  Testing GR, neutron star EOS, BSM effects...

ANATOMY OF GRAVITATIONAL WAVE SIGNAL



[Picture: Antelis, Moreno, 1610.03567]

Inspiral Post-Newtonian / Post-Minkowskian / EOB
Merger Numerical relativity / EOB resummation
Ringdown Perturbative quasi-normal modes

POST-NEWTONIAN EXPANSION

Virial theorem $G \sim P^2$. Hamiltonian in c.o.m. frame:

$$\frac{H}{\mu} = \boxed{\frac{P^2}{2} - \frac{Gm}{R}} \quad \text{Newton } \sim \mathcal{O}(G) \qquad \begin{array}{l} m = m_A + m_B, \quad \nu = \mu/M = P \cdot \hat{R} \\ \mu = m_A m_B/m \end{array}$$

$$\boxed{+ \frac{1}{c^2}\left\{-\frac{P^4}{8} + \frac{3\nu P^4}{8} + \frac{Gm}{R}\left(-\frac{P_R^2 \nu}{2} - \frac{3P^2}{2} - \frac{\nu P^2}{2}\right) + \frac{G^2 m^2}{2R^2}\right\}}$$

Einstein, Infeld, Hoffman, 1PN $\sim \mathcal{O}(G^2)$

1PN [Einstein, Infeld, Hoffman '38]. 2PN [Ohta *et al.*, '73]. 3PN [Jaranowski, Schaefer, '97; Damour, Jaranowski, Schaefer, '97; Blanchet, Faye, '00; Damour, Jaranowski, Schaefer, '01] 4PN [Damour, Jaranowski, Schäfer, Bernard, Blanchet, Bohe, Faye, Marsat, Marchand, Foffa, Sturani, Mastrolia, Sturm, Porto, Rothstein...] 5PN static [Foffa, Mastrolia, Sturani, Sturm, Bodabilla, '19; Blümlein, Maier, Marquard, '19] 5PN approximate [Bini, Damour, Geralico, '19]

POST-MINKOWSKIAN EXPANSION

Bound orbit: $GM/r \sim v^2$. Hyperbolic orbit / scattering: expand with $GM/r \leq v^2 \sim 1$. [Bertotti, Kerr, Plebanski, Portilla, Westpfahl, Goller, Bel, Damour, Deruelle, Ibanez, Martin, Ledvinka, Schaefer, Bicak...]

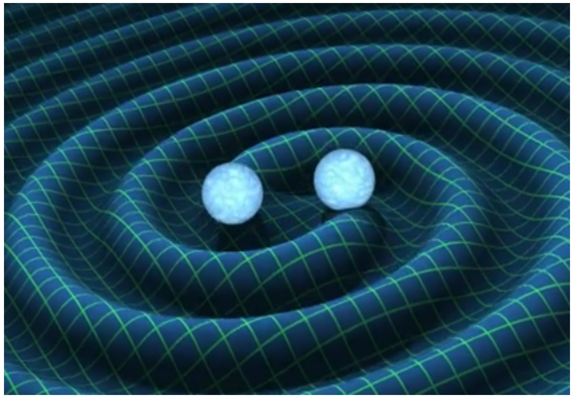
	0PN		**1PN**		**2PN**		**3PN**		**4PN**		**5PN**		**6PN**		**7PN**			
1PM	$(1$	$+$	v^2	$+$	v^4	$+$	v^6	$+$	v^8	$+$	v^{10}	$+$	v^{12}	$+$	v^{14}	$+$	$\dots)$	G^1
2PM			$(1$	$+$	v^2	$+$	v^4	$+$	v^6	$+$	v^8	$+$	v^{10}	$+$	v^{12}	$+$	$\dots)$	G^2
3PM					$(1$	$+$	v^2	$+$	v^4	$+$	v^6	$+$	v^8	$+$	v^{10}	$+$	$\dots)$	G^3
4PM							$(1$	$+$	v^2	$+$	v^4	$+$	v^6	$+$	v^8	$+$	$\dots)$	G^4
5PM									$(1$	$+$	v^2	$+$	v^4	$+$	v^6	$+$	$\dots)$	G^5
6PM											$(1$	$+$	v^2	$+$	v^4	$+$	$\dots)$	G^6
												$\vdots$						

Our new 3PM result: [Bern, Cheung, Roiban, Shen, Solon, MZ, arXiv: 1901.04424 (PRL), arXiv:1908.01493 (JHEP)]

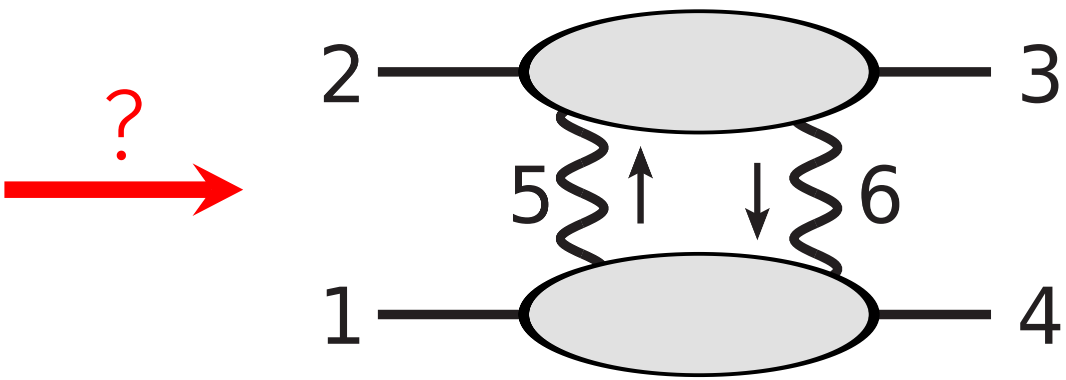
HOW QFT HELPS

Hierarchy of scales in bound state systems: $R_s \leq r \leq \lambda$

effective field theory: [NRGR: Goldberger, Rothstein, '04; Porto, '06; relativistic formulation: Damgaard, Haddad, Helset, '19]



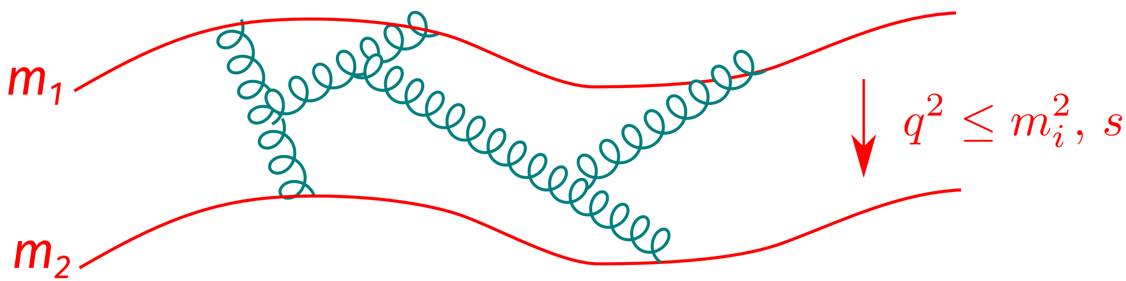
[picture: LIGO]



Manifest *gauge invariance* through scattering amplitudes, with carefully defined *classical limit* [Iwasaki '71; Gupta, Radford, '79; Donoghue, '94; Holstein, Donoghue, '04; Neill, Rothstein, '13; Vaidya, '14; Kosower, Maybee, O'Connell, '18; Cheung, Rothstein, Porto, '18; Kälin, Porto, '19...]

WHICH DIAGRAMS CONTRIBUTE CLASSICALLY?

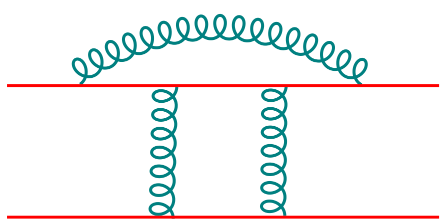
Rule 1: each loop must have a matter propagator.

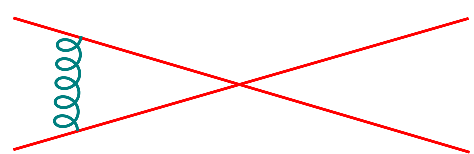


Inspiration from **NRGR** [Goldberger, Rothstein, '04]: Einstein-Hilbert action coupled to worldline sources

$$S = S_{EH} + S_{pp}, \quad S_{pp} = -\sum_a m_a \int d\tau_a + \sum_a c_R^{(a)} \int d\tau_a R\left(x_a\right) + \ldots$$

Rule 2: no "mushrooms", i.e. graviton connecting same matter.





Rule 3: no contact integrals with matter lines touching.

PERTURBATIVE GRAVITY

Einstein-Hilbert action

$$S_{\text{EH}} = \frac{1}{2\kappa^2}\int d^dx\sqrt{-g}R + S_{\text{matter}}$$

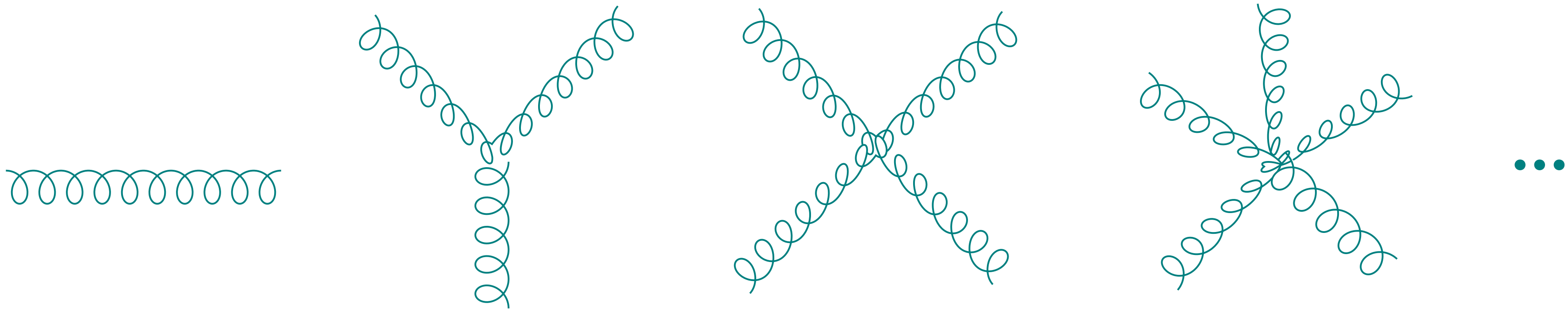
Expand around flat space

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$$

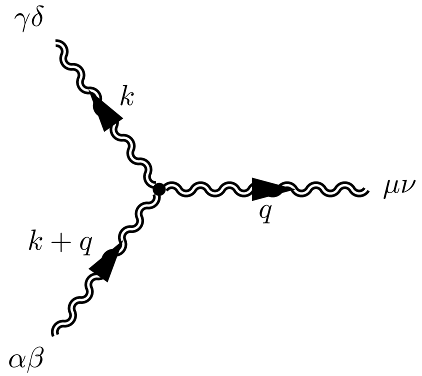
Schematic, index contractions omitted

$$S_{\text{EH}} = \int d^dx\left[h\partial^2 h + \kappa h^2\partial^2 h + \kappa^2 h^3\partial^2 h + \kappa^3 h^4\partial^2 h + \dots\right]$$

[e.g. Elvang, Huang, '13]



PERTURBATIVE GRAVITY



$$
\begin{aligned}
\tau^{\mu\nu}_{\alpha\beta,\gamma\delta}(k,q) = -\frac{i\kappa}{2}\Big\{ & P_{\alpha\beta,\gamma\delta}\Big[k^\mu k^\nu + (k+q)^\mu(k+q)^\nu + q^\mu q^\nu - \frac{3}{2}\eta^{\mu\nu}q^2\Big] \\
& +2q_\lambda q_\sigma\Big[I^{\lambda\sigma,}{}_{\alpha\beta}I^{\mu\nu,}{}_{\gamma\delta} + I^{\lambda\sigma,}{}_{\gamma\delta}I^{\mu\nu,}{}_{\alpha\beta} \\
& \qquad - I^{\lambda\mu,}{}_{\alpha\beta}I^{\sigma\nu,}{}_{\gamma\delta} - I^{\sigma\nu,}{}_{\alpha\beta}I^{\lambda\mu,}{}_{\gamma\delta}\Big] \\
& +\Big[q_\lambda q^\mu(\eta_{\alpha\beta}I^{\lambda\nu,}{}_{\gamma\delta}+\eta_{\gamma\delta}I^{\lambda\nu,}{}_{\alpha\beta}) + q_\lambda q^\nu(\eta_{\alpha\beta}I^{\lambda\mu,}{}_{\gamma\delta}+\eta_{\gamma\delta}I^{\lambda\mu,}{}_{\alpha\beta}) \\
& \quad - q^2(\eta_{\alpha\beta}I^{\mu\nu,}{}_{\gamma\delta}+\eta_{\gamma\delta}I^{\mu\nu,}{}_{\alpha\beta}) - \eta^{\mu\nu}q^\lambda q^\sigma(\eta_{\alpha\beta}I_{\gamma\delta,\lambda\sigma}+\eta_{\gamma\delta}I_{\alpha\beta,\lambda\sigma})\Big] \\
& +\Big[2q^\lambda\Big(I^{\sigma\nu,}{}_{\gamma\delta}I_{\alpha\beta,\lambda\sigma}k^\mu + I^{\sigma\mu,}{}_{\gamma\delta}I_{\alpha\beta,\lambda\sigma}k^\nu \\
& \qquad - I^{\sigma\nu,}{}_{\alpha\beta}I_{\gamma\delta,\lambda\sigma}(k+q)^\mu - I^{\sigma\mu,}{}_{\alpha\beta}I_{\gamma\delta,\lambda\sigma}(k+q)^\nu\Big) \\
& \quad +q^2(I^{\sigma\mu,}{}_{\alpha\beta}I_{\gamma\delta,\sigma}{}^\nu + I_{\alpha\beta,\sigma}{}^\nu I^{\sigma\mu,}{}_{\gamma\delta}) \\
& \quad +\eta^{\mu\nu}q^\lambda q_\sigma(I_{\alpha\beta,\lambda\rho}I^{\rho\sigma,}{}_{\gamma\delta} + I_{\gamma\delta,\lambda\rho}I^{\rho\sigma,}{}_{\alpha\beta})\Big] \\
& +\Big[(k^2+(k+q)^2)\Big(I^{\sigma\mu,}{}_{\alpha\beta}I_{\gamma\delta,\sigma}{}^\nu + I^{\sigma\nu,}{}_{\alpha\beta}I_{\gamma\delta,\sigma}{}^\mu - \frac{1}{2}\eta^{\mu\nu}P_{\alpha\beta,\gamma\delta}\Big) \\
& \quad -((k+q)^2\eta_{\alpha\beta}I^{\mu\nu,}{}_{\gamma\delta} + k^2\eta_{\gamma\delta}I^{\mu\nu,}{}_{\alpha\beta})\Big]\Big\}
\end{aligned}
$$

Background field gauge vertex [Holstein, Ross, 0802.0716]

About 100 terms in 3-graviton vertex!

Quickly grows out of control with more loops & legs.

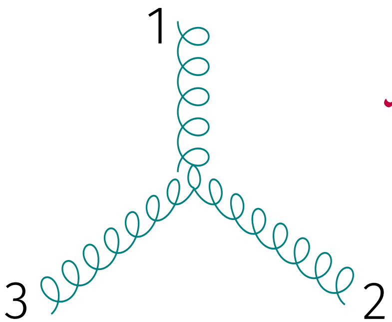
Double copy & Generalized Unitarity to the rescue

Double copy: gravity from YM

Generalized Unitarity: Loops from trees

GRAVITY = (YANG MILLS)^2

On-shell 3-point amplitudes in Yang-Mills & gravity:



$$\mathcal{A}_3(1^-2^-3^+) = \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}, \quad \mathcal{M}_3(1^-2^-3^+) = \frac{\langle 12\rangle^6}{\langle 23\rangle^2\langle 31\rangle^2}$$

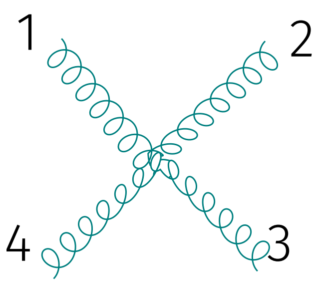
Square!

In 4D, $(A^+)^2 \sim h^{++}$, $(A^-)^2 \sim h^{--}$

$A^+A^- \sim$ ~~dilaton, axion~~

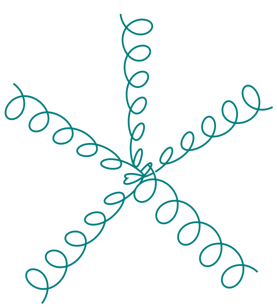
[Johansson, Ochirov, '14; Luna, Nicholson, O'Connel, White, '17]

Kawai-Lewellen-Tye (KLT) relations from string theory:



$$M_4^{\text{tree}}(1,2,3,4) = -is_{12}\, A_4^{\text{tree}}(1,2,3,4)\, A_4^{\text{tree}}(1,2,4,3)$$

$$\mathcal{A}_4^{\text{tree}} \equiv g^2 \sum_{\sigma\in S_4/Z_4} A_4^{\text{tree}}(\sigma_1,\sigma_2,\sigma_3,\sigma_4)\, \text{tr}\left(T^{a_{\sigma_1}}T^{a_{\sigma_2}}T^{a_{\sigma_3}}T^{a_{\sigma_4}}\right)$$



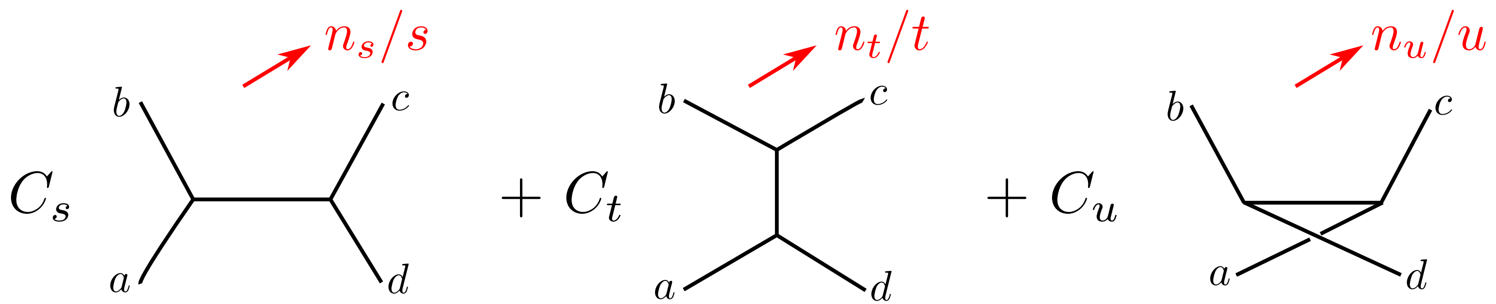
$$M_5^{\text{tree}}(1,2,3,4,5) = is_{12}s_{34}\, A_5^{\text{tree}}(1,2,3,4,5)\, A_5^{\text{tree}}(2,1,4,3,5) + is_{13}s_{24}\, A_5^{\text{tree}}(1,3,2,4,5)\, A_5^{\text{tree}}(3,1,4,2,5)$$

$$\vdots$$

COLOR KINEMATIC DUALITY

D dimensions: easier to get nice analytic integrands from **color-kinematics duality**, i.e. double copy construction. [Bern, Carrasco, Johansson, '08]

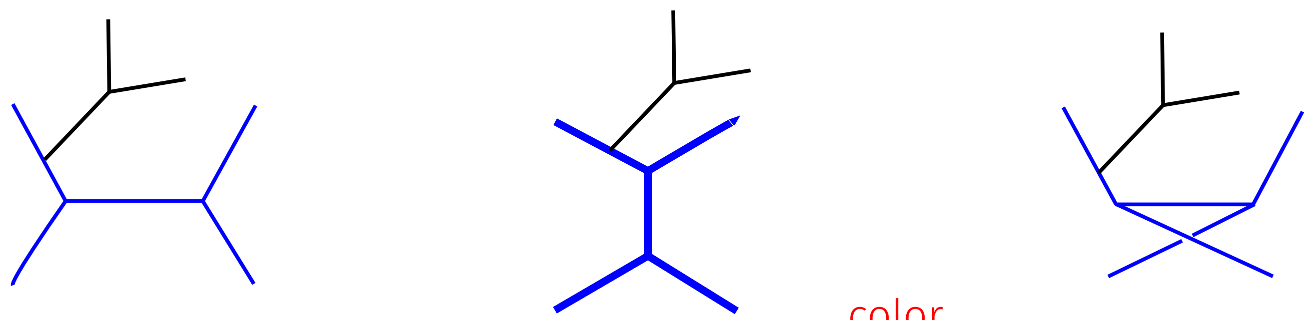
Let's look at the 4-gluon amplitude, with 4-point vertex "blown up" to 3-pt.



$$C_s = f^{abf} f^{cdf}, \quad C_t = f^{bcf} f^{daf}, \quad C_u = f^{afc} f^{dbf},$$

Jacobi identity: $C_s + C_t + C_u = 0.$ **Surprise:** $n_s + n_t + n_u = 0.$

COLOR KINEMATIC DUALITY



color

kinematics

BCJ form of integrand: $A = \sum_{i \in \text{diags}} \frac{C_i \, n_i}{D_i}, \quad n_i + n_j + n_k = 0$

Propagaators

whenever $C_i + C_j + C_k = 0$

Gravity integrand: $M = \sum_{i \in \text{diags}} \frac{n_i \, \tilde{n}_i}{D_i}$, both $n_i, \tilde{n}_i$ in BCJ form

Generalized gauge transformation: $n_l \to n_l + \delta \cdot D_l, \quad l = i, j, k$

A invariant ✓

Gauge invariance

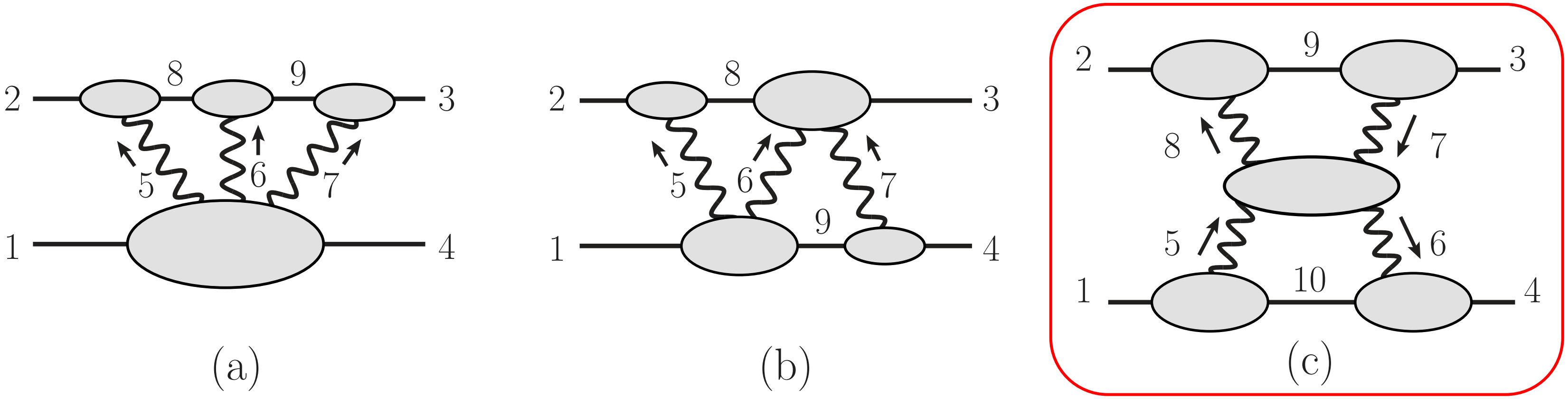
M invariant ✓

diffeomorphism invariance

TWO-LOOP CUTS

[Bern, Cheung, Roiban, Shen, Solon, MZ, arXiv:1908.01493 (JHEP)]

$$S = \int d^D x \sqrt{g} \left[-\frac{1}{2} R + \frac{1}{2} \sum_{i=1,2} \left(D^\mu \phi_i D_\mu \phi_i - m_i \phi_i^2 \right) \right]$$



From KLT: $M_4^{\rm tree}(1,2,3,4) = -i s_{12} \mathcal{A}_4^{\rm tree}(1,2,3,4) A_4^{\rm tree}(1,2,4,3)$.

$$C_{\rm GR}^{(c)} = -i \left\{ 2t^2 m_1^4 m_2^4 + \frac{1}{t^6} \left[Tr[(\not 7 \not 2 \not 8 \not 6 \not 1 \not 5)^4] + (7 \leftrightarrow 8) \right] \right\} \left(\frac{1}{(k_5 - k_8)^2} + \frac{1}{(k_6 + k_8)^2} \right)$$

Very compact expression at 2 loops. Higher loops within reach!

TWO-LOOP INTEGRAND

Cuts *merged* into an integrand with diagrams & numerators:

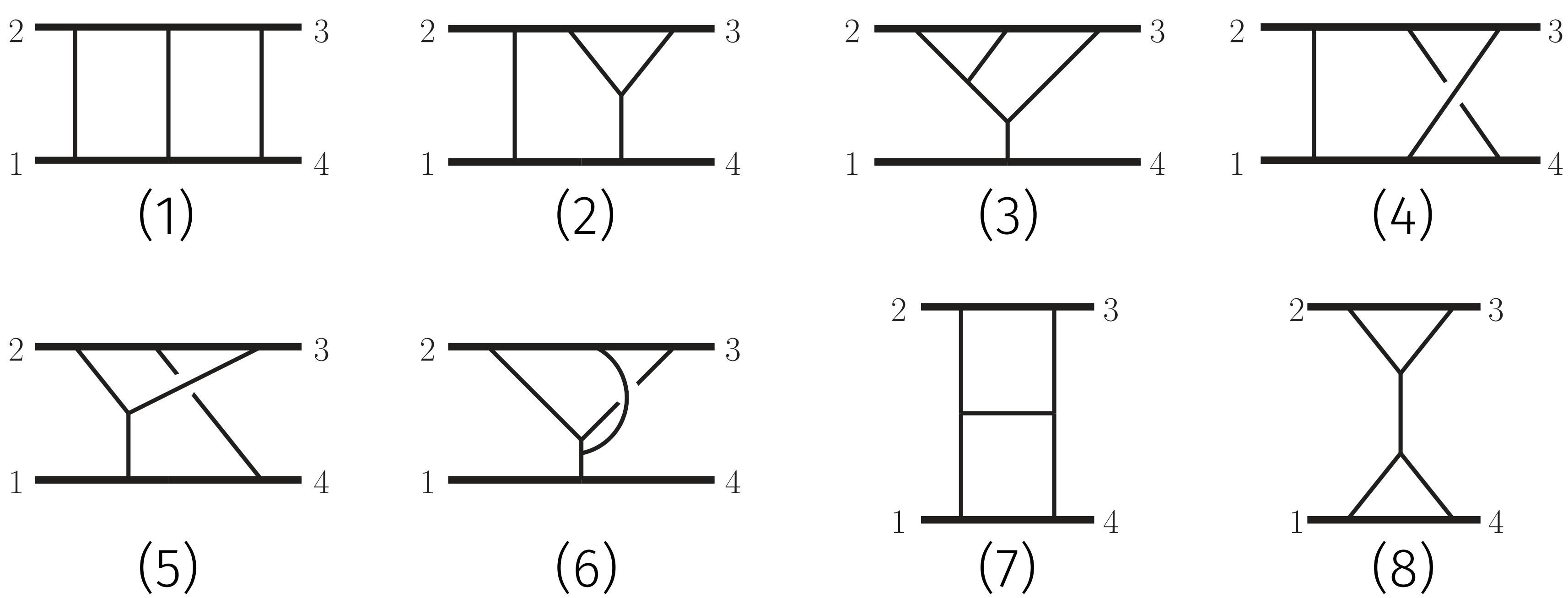
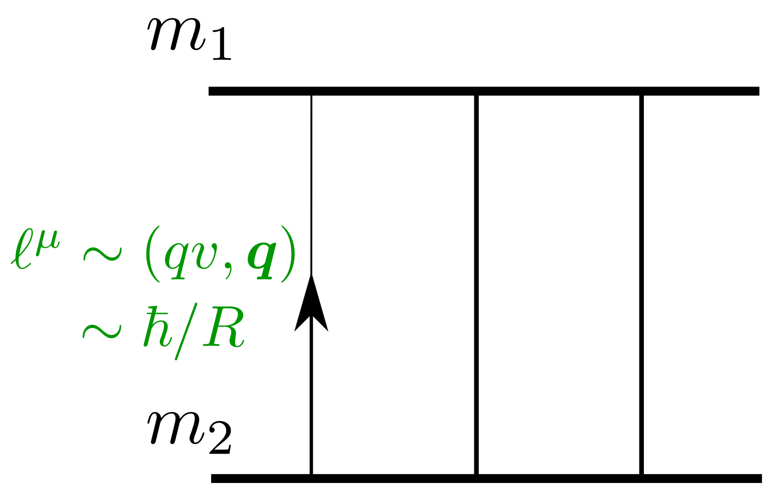


Diagram symmetries imposed. $\sim$ 90KB file.

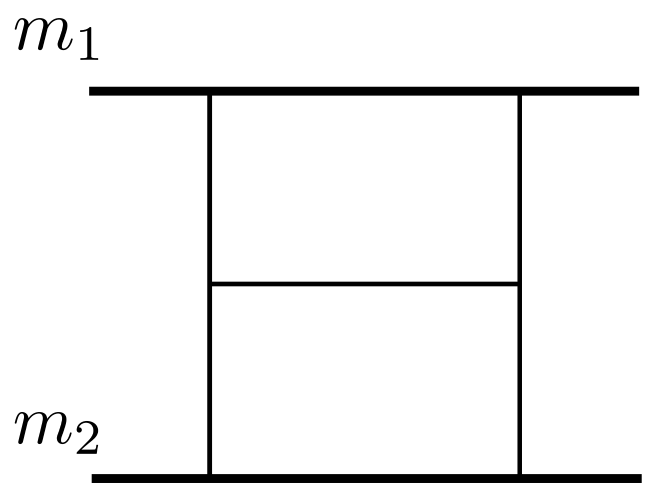
4 and *D* dimensional results agree, up to "μ" terms with no classical effects.

INTEGRATING THE AMPLITUDE

- $m_1 \neq m_2$, even planar master integrals unknown!



$m_1 = m_2$ Smirnov, '01; Lower topologies: Henn & Smirnov, '13; Duhr, Amplitudes '18; Heller, von Manteuffel, Schabinger, '19



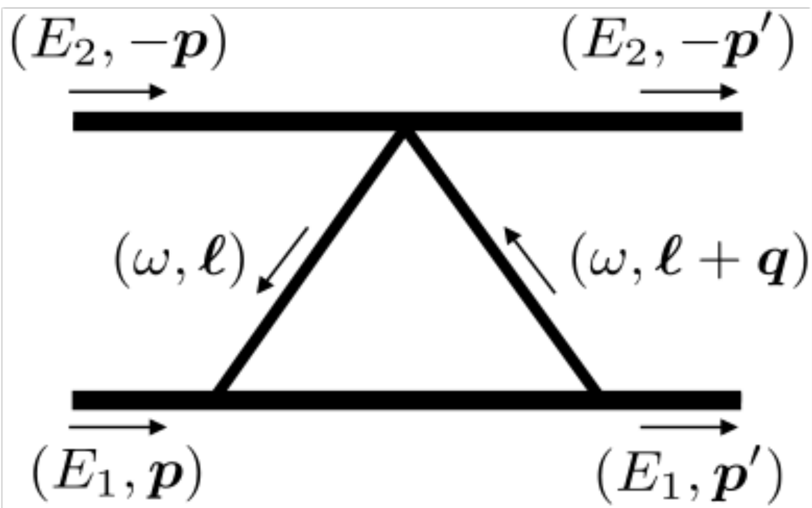
$m_1 = m_2$ Bianchi, Leoni, 1612.05609

- Simplification 1: expand in small $q \sim \hbar/R \leq m_i, \sqrt{s}$.
- Simplification 2: Expand in $v \ll 1$ from potential region. ($\int d^4\ell$) localized on $+$ve energy matter poles.

NR INTEGRATION / VELOCITY EXPANSION

Plan: Series expansion around static limit, then resum by matching to simple functions. One-loop triangle toy example:

Step 1: determine integrand in potential region $\ell^0 = \omega \leq |\boldsymbol{l}| \sim |\boldsymbol{q}|$



$$\mathcal{I}_{\mathrm{T}} = \frac{1}{(E_1+\omega)^2 - (\boldsymbol{p}+\boldsymbol{l})^2 - m_1^2} \times \frac{1}{\omega^2 - \boldsymbol{l}^2} \frac{1}{\omega^2 - (\boldsymbol{l}+\boldsymbol{q})^2}$$

$$= \boxed{\frac{1}{\omega - \omega_{P_1}}} \frac{1}{2m_1 \boldsymbol{l}^2 (\boldsymbol{l}+\boldsymbol{q})^2} + \ldots$$

need prescription for $d\omega$ integration

$$\frac{1}{(E_1+\omega)^2 - (\boldsymbol{p}+\boldsymbol{l})^2 - m_1^2} = \frac{1}{(\omega - \omega_{P_1})(\omega - \omega_{A_1})},$$

$$\omega_{P_1}, \omega_{A_1} = -E_1 \pm \sqrt{E_1^2 + 2\boldsymbol{p}\boldsymbol{l} + \boldsymbol{l}^2}\,. \quad \omega_{P_1} \ll |\boldsymbol{l}|,\ \omega_{A_1} \approx -2m_1$$

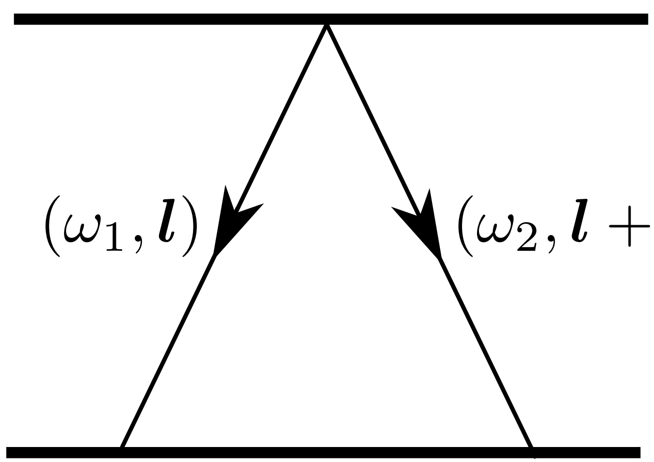
NR INTEGRATION / VELOCITY EXPANSION

Step 2: Energy integration, keeping only residues from $+$ve energy matter poles.

$$\frac{1}{2\pi}\int d\omega_1 \frac{1}{\omega_1 - \omega_{P_1} + i0} = \frac{1}{2\pi}\int d\omega_1 \frac{1}{2}\left(\frac{1}{\omega_1 - \omega_{P_1} + i0} + \frac{1}{-\omega_1 - \omega_{P_1} + i0}\right)$$

$$= \frac{1}{2\pi}\int d\omega_1 \frac{1}{2}(-2\pi i)\delta(\omega_1) = -\frac{i}{2}$$

Higher loops: symmetrize over $\omega_1, \omega_2, \ldots, \omega_n$.

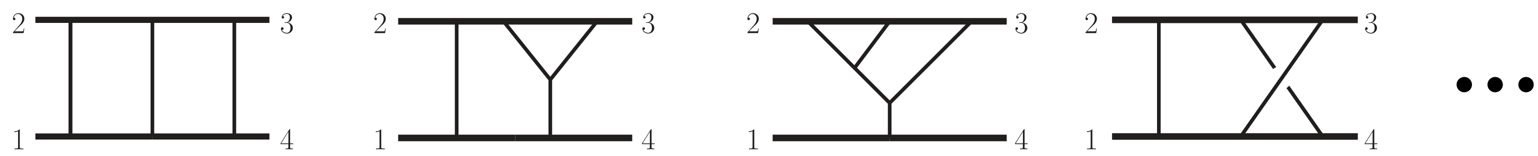


Step 3: Spatial integration at small $|q|$.

$$\int \frac{d^3\boldsymbol{l}}{(2\pi)^3}\left(-\frac{i}{2}\right)\frac{1}{2m_1\,\boldsymbol{l}^2(\boldsymbol{l}+\boldsymbol{q})^2} = -\frac{i}{32m_1|\boldsymbol{q}|}\,.$$

Only need 3D propagator integrals (known to at least 4 loops) + *divergent integrals from box-type diagrams.*

RESULT FOR AMPLITUDE (PN EXPANSION)



(1) (2) (3) (4)

Sample result:

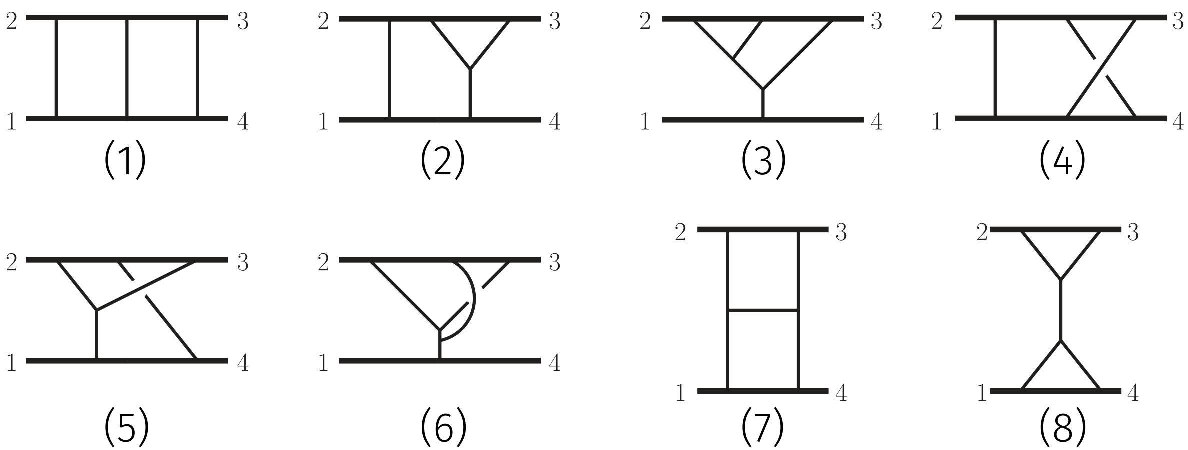
Unevaluated divergent integral; cancels in EFT matching

$$
\begin{aligned}
4 = & \frac{(2\sigma^2-1)(4\sigma^2-1)\, m_1^4 m_2^6}{8E_2 E} \int \frac{d^{D-1}\ell}{(2\pi)^{D-1}} \frac{1}{\ell^2 |\ell+q| \left(\ell^2+2p\ell\right)} \left[1+\frac{p^2}{2E_2^2}+\frac{3p^4}{8E_2^4}+\frac{5p^6}{16E_2^6}+\cdots\right] \\
& + \frac{\sigma m_1^3 m_2^5 (4\sigma^2-1)}{48\pi^2 E_2^2} \left[1+\frac{p^2}{E_2^2}+\frac{p^4}{E_2^4}+\frac{p^6}{E_2^6}+\frac{p^8}{E_2^8}+\frac{p^8}{E_2^{10}}+\cdots\right] \\
& - \frac{m_1^4 m_2^6 (2\sigma^2-1)(4\sigma^2-1)}{128\pi^2 E_2^3 E} \left[1+\frac{5p^2}{4E_2^2}+\frac{11p^4}{8E_2^4}+\frac{93p^6}{64E_2^6}+\frac{193p^8}{128E_2^8}+\frac{793p^{10}}{512E_2^{10}}+\cdots\right] + \{1 \leftrightarrow 2\}
\end{aligned}
$$

$$
\sigma \equiv (p_1 \cdot p_2)/(m_1 m_2)
$$

- Let's fit the series to simple functions… See next slide.
- Better still, can we do relativistic integration, and with ε dependence?

RESULT FOR AMPLITUDE (PM EXPANSION)



Fit PN velocity series to simple functions.

Computer algebra automation: [Blümlein, Marquard, Schäfer, Schneider, '19]

$$m = m_1 + m_2,\ \nu = \frac{m_1 m_2}{m^2},\ E = E_1 + E_2,\ \xi = \frac{E_1 E_2}{E^2},\ \gamma = \frac{E}{m},\ \sigma = \frac{p_1 \cdot p_2}{m_1 m_2}$$

$1/r^3$ potential after F.T.

$$\mathcal{M}_3 = \frac{\pi G^3 \nu^2 m^4 \ln \boldsymbol{q}^2}{6\gamma^2 \xi}\Bigg[3 - 6\nu + 206\nu\sigma - 54\sigma^2 + 108\nu\sigma^2 + 4\nu\sigma^3 - \frac{48\nu\left(3 + 12\sigma^2 - 4\sigma^4\right)\operatorname{arcsinh}\sqrt{\frac{\sigma-1}{2}}}{\sqrt{\sigma^2-1}} - \frac{18\nu\gamma\left(1-2\sigma^2\right)\left(1-5\sigma^2\right)}{(1+\gamma)(1+\sigma)}\Bigg]$$
$$+ \frac{8\pi^3 G^3 \nu^4 m^6}{\gamma^4 \xi}\Bigg[3\gamma\left(1-2\sigma^2\right)\left(1-5\sigma^2\right)\int \frac{d^{D-1}\boldsymbol{\ell}}{(2\pi)^{D-1}} \frac{1}{\boldsymbol{\ell}^2|\boldsymbol{\ell}+\boldsymbol{q}|(\boldsymbol{\ell}^2 + 2\boldsymbol{p}\boldsymbol{\ell})}$$
$$- 32 m^2 \nu^2 \left(1-2\sigma^2\right)^3 \int \frac{d^{D-1}\boldsymbol{\ell}_1}{(2\pi)^{D-1}} \frac{d^{D-1}\boldsymbol{\ell}_2}{(2\pi)^{D-1}} \frac{1}{\boldsymbol{\ell}_1^2(\boldsymbol{\ell}_2-\boldsymbol{\ell}_1)^2(\boldsymbol{\ell}_2+\boldsymbol{q})^2(\boldsymbol{\ell}_1^2+2\boldsymbol{p}\boldsymbol{\ell}_1)(\boldsymbol{\ell}_2^2+2\boldsymbol{p}\boldsymbol{\ell}_2)}\Bigg]$$

Most complicated function, seen in **relativistic integration** of diagram 7.

POTENTIAL FROM EFT

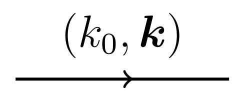
[Cheung, Rothstein, Solon, 1808.02489]

Lagrangian: two non-relativistic scalars

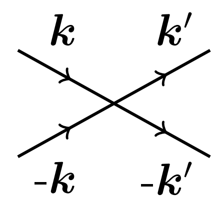
$$\mathcal{L} = \sum_{i=1,2} \int_{\boldsymbol{k}} \phi_i^\dagger(-\boldsymbol{k}) \left(i\partial_t - \sqrt{\boldsymbol{k}^2 + m_i^2} \right) \phi_i(\boldsymbol{k})$$

$$- \int_{\boldsymbol{k},\boldsymbol{k}'} V(\boldsymbol{k},\boldsymbol{k}') \phi_1^\dagger(\boldsymbol{k}')\phi_1(\boldsymbol{k})\phi_2^\dagger(-\boldsymbol{k}')\phi_2(-\boldsymbol{k})$$

Feynman rules:

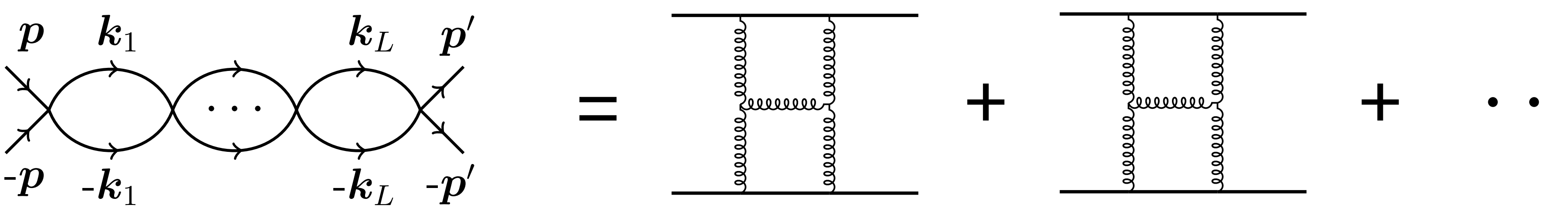


$$= \frac{i}{k_0 - \sqrt{\boldsymbol{k}^2 + m_{A,B}^2} + i0},$$



$$= -iV(\boldsymbol{k}, \boldsymbol{k}'),$$

Determines V from **Matching:** EFT amplitude = full theory amplitude.



simple 3D triangles

energy integration leaves 3D integrals

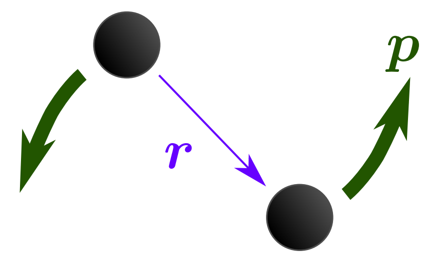
Alternative QM treatment: [Cristofoli, Bjerrum-Bohr, Damgaard, Vanhove, '19]

RESULT: 3PM CONSERVATIVE POTENTIAL

[Bern, Cheung, Roiban, Shen, Solon, MZ '19]

$$H^{3\mathrm{PM}}(\boldsymbol{p},\boldsymbol{r}) = \sqrt{\boldsymbol{p}^2+m_1^2}+\sqrt{\boldsymbol{p}^2+m_2^2}+V^{3\mathrm{PM}}(\boldsymbol{p},\boldsymbol{r})$$

$$V^{3\mathrm{PM}}(\boldsymbol{p},\boldsymbol{r}) = c_1(\boldsymbol{p}^2)\left(\frac{G}{|\boldsymbol{r}|}\right)+c_2(\boldsymbol{p}^2)\left(\frac{G}{|\boldsymbol{r}|}\right)^2+c_3(\boldsymbol{p}^2)\left(\frac{G}{|\boldsymbol{r}|}\right)^3$$



$$m = m_1+m_2,\ \nu=\frac{m_1m_2}{m^2},\ E=E_1+E_2,\ \xi=\frac{E_1E_2}{E^2},\ \gamma=\frac{E}{m},\ \sigma=\frac{p_1\cdot p_2}{m_1m_2}$$

$$c_1=\frac{\nu^2m^2}{\gamma^2\xi}\left(1-2\sigma^2\right),\quad c_2=\frac{\nu^2m^3}{\gamma^2\xi}\left[\frac{3}{4}\left(1-5\sigma^2\right)-\frac{4\nu\sigma\left(1-2\sigma^2\right)}{\gamma\xi}-\frac{\nu^2(1-\xi)\left(1-2\sigma^2\right)^2}{2\gamma^3\xi^2}\right],$$

$$\begin{aligned}c_3=\frac{\nu^2m^4}{\gamma^2\xi}\Bigg[&\frac{1}{12}\left(3-6\nu+206\nu\sigma-54\sigma^2+108\nu\sigma^2+4\nu\sigma^3\right)-\frac{4\nu\left(3+12\sigma^2-4\sigma^4\right)\operatorname{arcsinh}\sqrt{\frac{\sigma-1}{2}}}{\sqrt{\sigma^2-1}}\\&-\frac{3\nu\gamma\left(1-2\sigma^2\right)\left(1-5\sigma^2\right)}{2(1+\gamma)(1+\sigma)}-\frac{3\nu\sigma\left(7-20\sigma^2\right)}{2\gamma\xi}+\frac{2\nu^3(3-4\xi)\sigma\left(1-2\sigma^2\right)^2}{\gamma^4\xi^3}\\&-\frac{\nu^2\left(3+8\gamma-3\xi-15\sigma^2-80\gamma\sigma^2+15\xi\sigma^2\right)\left(1-2\sigma^2\right)}{4\gamma^3\xi^2}+\frac{\nu^4(1-2\xi)\left(1-2\sigma^2\right)^3}{2\gamma^6\xi^4}\Bigg].\end{aligned}$$

Validation

Bern, Cheung, Roiban, Shen, Solon, MZ '19

New 3PM Hamiltonian

Canonical transformation

$$(\boldsymbol{r},\boldsymbol{p}) \to (\boldsymbol{R},\boldsymbol{P}) = (A\boldsymbol{r} + B\boldsymbol{p}, C\boldsymbol{p} + D\boldsymbol{r})$$

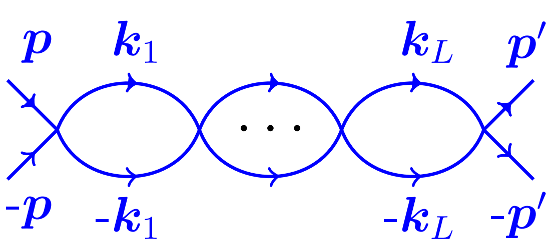
$$A = 1 - \frac{Gm\nu}{2|\boldsymbol{r}|} + \cdots\,, \quad B = \frac{G(1-2/\nu)}{4m|\boldsymbol{r}|}\boldsymbol{p}\cdot\boldsymbol{r} + \cdots$$

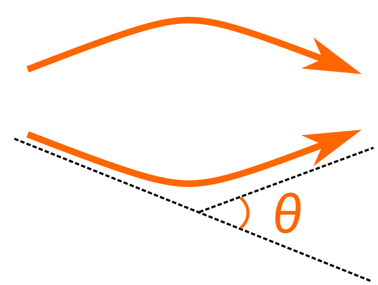
Jaranowski, Schäfer, 1508.01016; Bini, Damour, Geralico, 1909.02375

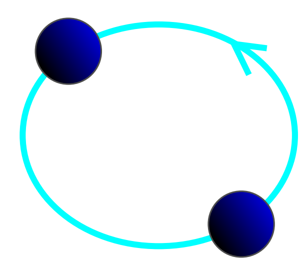
Overlap with 4PN, 5PN

***Gauge-invariant* quantities**

- Scattering amplitude
- Scattering angle
- Binding energy of quas-circular orbit

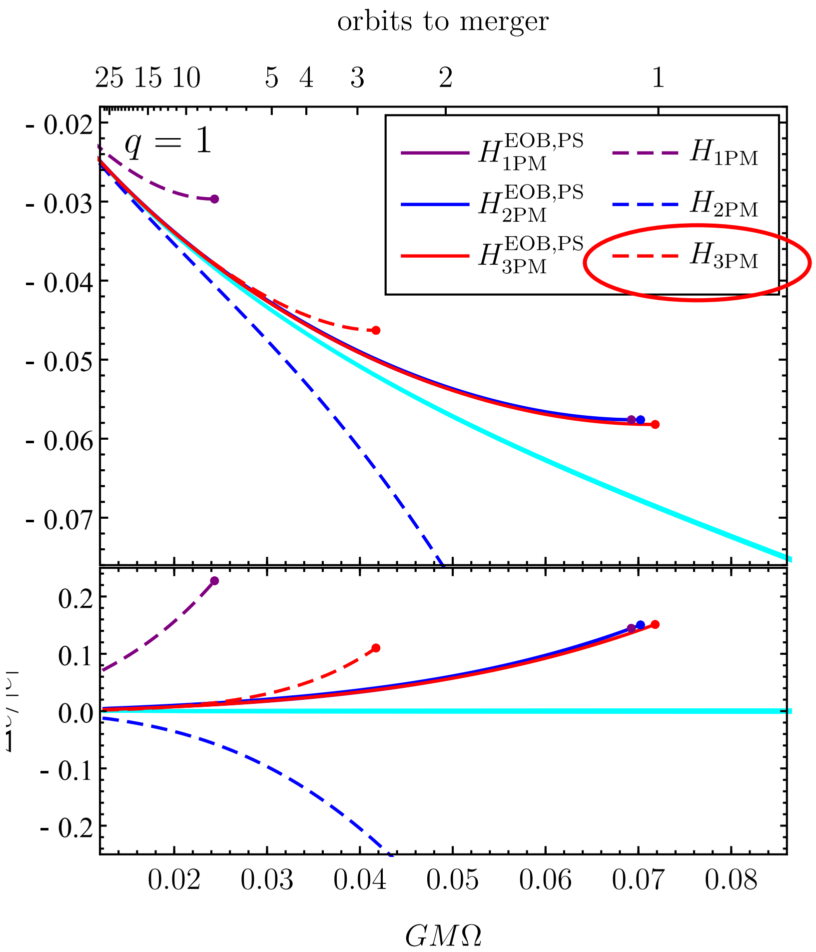






REAL WORLD PREDICTION - BINDING ENERGY

Comparison with state-of-the-art numerical relativity ("truth"), PN, and EOB resummation predictions. [Antonelli, Buonanno, Steinhoff, Vines, '19]

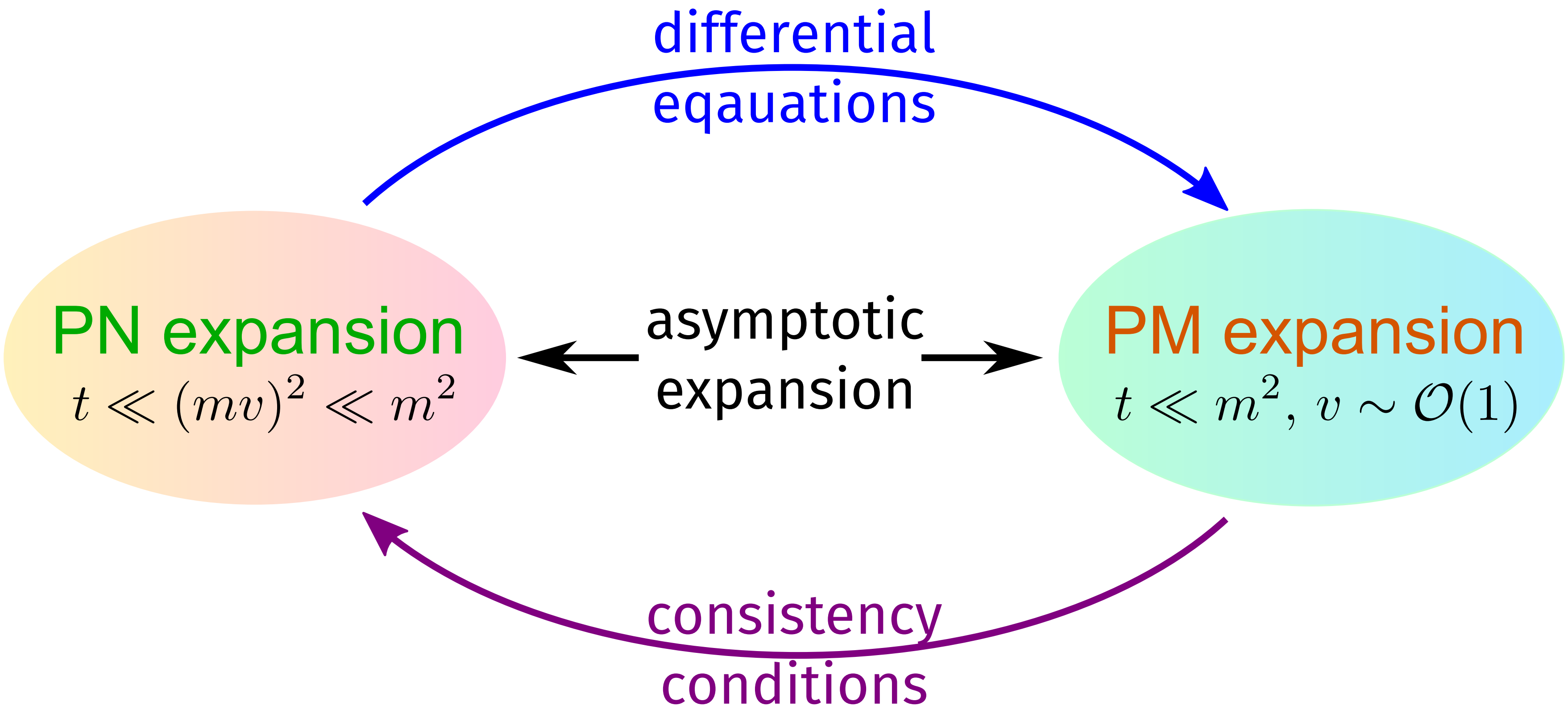


- ***Clear improvement*** over lower PM orders.
- Not reaching accuracy of 4PN.
- Hyperbolic orbit comparison desirable to assess PM approxmation.
- ***Wish list:*** 4 PM / 3 loops!

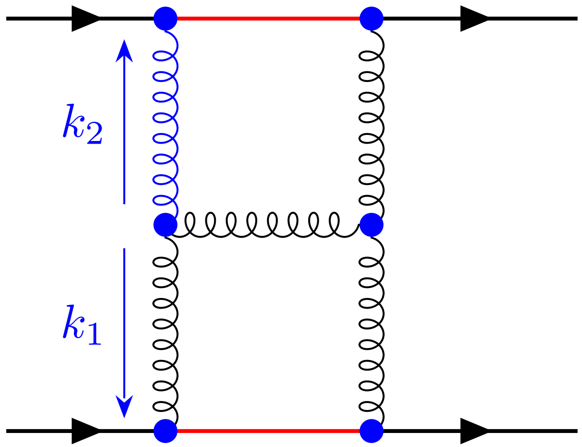
RELATIVISTIC INTEGRATION

- Velocity resummation non-trivial! We can recover exact v dependence of amplitude, but not always every diagram.

- Desirable to perform relativistic integration. Powerful method: ***differential equations*** Kotikov, '91; Bern, Dixon, Kosower, '94; Remiddi, '97; Gehrmann, Remiddi, 99



RELATIVISTIC INTEGRATION: EXPANSION



Perform soft expansion:

$|k_1| \sim |k_2| \sim |q| \ll m_1, m_2, \sqrt{s}$

Method of regions:
Beneke, Smirnov, '98

Heavy quark effective theory:
Georgi, Eichten, Hill, Isgur, Wise, Shifman...

Heavy BH effective theory:
Damgaard, Haddad, Helset, '19

Kinematics

$p_1^2 = m_1^2 + t/4,\ p_2^2 = m_2^2 + t/4$

$u_1^2 = u_2^2 = 1,\ u_1 \cdot u_2 = \sigma, \quad u_1 \cdot q = u_2 \cdot q = 0, q^2 = t$

only intrinsic scale of integral

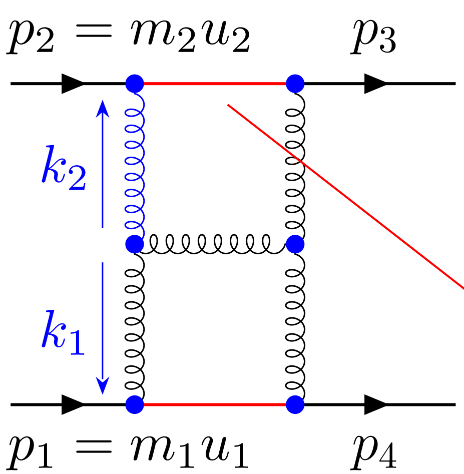
Expand matter propagators

$(k_2 + p_2)^2 - m_2^2 \approx 2 m_2 u_2 \cdot k_2 + i0$

$(k_1 + p_1)^2 - m_1^2 \approx 2 m_1 u_1 \cdot k_1 + i0$

Soft integrals are functions of the dimensionless parameter σ

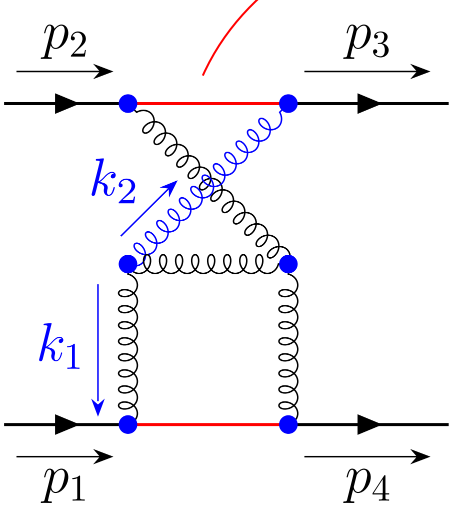
LOCALIZATION ON MATTER POLES



Classical picture arises from nontrivial cross-talk between planar and nonplanar diagrams.

$$(k_2 + p_2)^2 - m_2^2 \approx 2m_2 u_2 \cdot k_2 + i0$$

$$(k_2 - p_3)^2 - m_2^2 \approx -2m_2 u_2 \cdot k_2 + i0$$

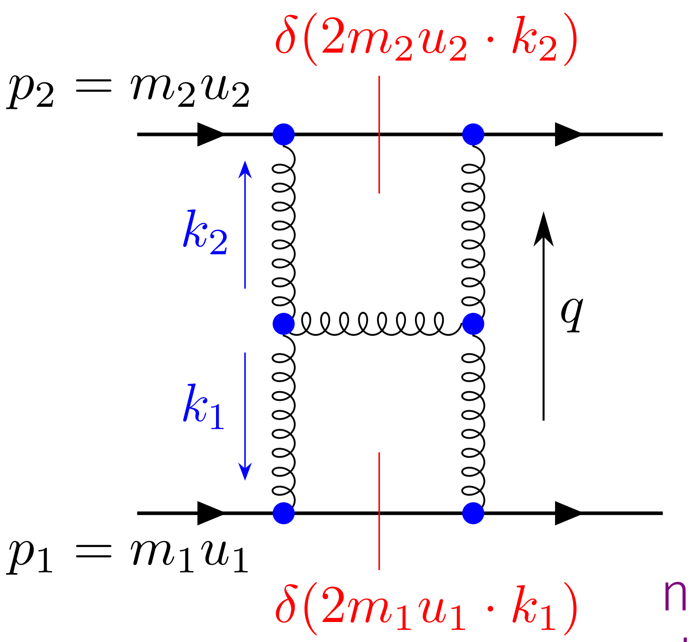


$$\frac{1}{2m_2 u_2 \cdot k_2 + i0} + \frac{1}{-2m_2 u_2 \cdot k_2 + i0} = -2\pi i\, \delta(2m_2 u_2 \cdot k_2)$$

Let's calculate the sum = **cut integrals** instead of individual ones!

cut integrals in other contexts: e.g. [Primo, Tancredi, '16, '17]

RELATIVISTIC INTEGRATION 1: CUT INTEGRALS



Ita, Ruf, MZ, in progress, simplified from Bern, Cheung, Roiban, Shen, Solon, MZ, '19

$$u_1\cdot q = u_2\cdot q = 0$$

$$u_1^2 = u_2^2 = 1,\ u_1\cdot u_2 = \sigma,\ q^2 = t$$

nontrivial dependence on this dimensionless variable

only intrinsic scale of integral

$$F_{p_1,p_2,\ldots,p_9} \equiv \int d^dk_1 d^dk_2 \frac{(k_1\cdot u_2)^{p_1}(k_2\cdot u_1)^{p_2}\delta^{(p_3)}(2u_1\cdot k_1)\,\delta^{(p_4)}(2u_2\cdot k_2)}{\left(k_1^2\right)^{p_5}\left((k_1+q)^2\right)^{p_6}\left(k_2^2\right)^{p_7}\left((k_2-q)^2\right)^{p_8}\left((k_1+k_2)^2\right)^{p_9}},$$

$\sim |q|^{2d+p_1+p_2-p3-p4-2p_5-\cdots-2p_9}$ × *Dimensionless function of σ*

Even & odd powers decouple

Integration by parts reduces an infinite number of such integrals to **five master integrals**, in the even sector. Timing: < 1 minute, FIRE 6 [Smirnov '19]

METHOD OF DIFFERENTIAL EQUATIONS

- Many methods for evaluating "master integrals":
  - Parametric integration
  - Mellin-Barnes representation
  - Differential equations
- DEs especially powerful in ***canonical form*** [Henn, '13]

ε factorization symbol letters matrix of rational numbers

$$\frac{\partial}{\partial x}\vec{I} = \epsilon \sum_r \frac{\partial \log W_r}{\partial x} \mathbb{M}^{(r)} \cdot \vec{I}$$

kinematic variable pure UT master integrals

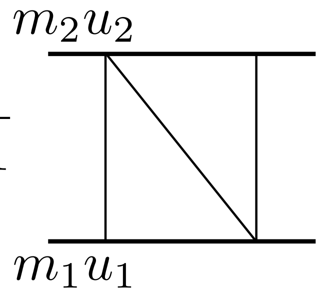
- Manifest ***logarithmic singularity*** & ***uniform transcendentality***, intimately related to dlog forms [Arkani-Hamed, Bourjaily, Cachazo, Trnka, 2010...]
- Solved iteratively order by order in ε, often as generalized polylogarithms. [Goncharov, Spradlin, Vergu, Volovich, 2010]

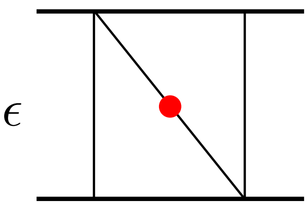
DIFFERENTIAL EQUATIONS WITH MATTER CUTS

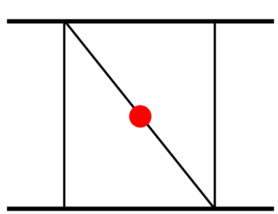
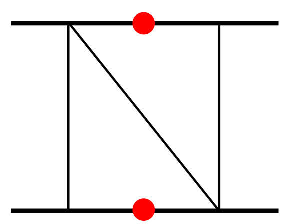
[Ita, Ruf, MZ, *in progress*]

static ↗ high-energy ↗

$$q^2 = t = -1,\ u_1^2 = u_2^2 = 1,\ u_1 \cdot u_2 = \sigma = \frac{1+x^2}{2x}, \quad 1 > x > 0$$

$I_1 = \sqrt{\sigma^2 - 1}$ 

, $I_2 = \sqrt{\sigma^2 - 1}/\epsilon$  ,

$I_3 = \sigma/\epsilon$  $-1/\epsilon^2$  , $d = 4 - 2\epsilon$

$$\frac{d}{dx}\begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} = \epsilon \left[\frac{d\log(1-x)}{dx}\mathbb{M}_1 + \frac{d\log x}{dx}\mathbb{M}_2 + \frac{d\log(1+x)}{dx}\mathbb{M}_3 \right] \cdot \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix},$$

↘ 3×3 matrix

Canonical form [Henn, '13] found by automated software `epsilon` [Prausa, '17]. See also `Fuchsia` [Gituliar, Magerya, '17], `CANONICA` [Meyer, '17].

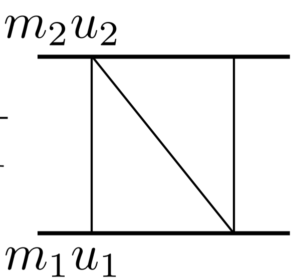
DIFFERENTIAL EQUATIONS WITH MATTER CUTS

[Ita, Ruf, MZ, *in progress*]

static ↗ high-energy ↗

$$u_1 \cdot u_2 = \sigma = \frac{1+x^2}{2x}, \qquad 1 > x > 0$$

$$I_1 = \sqrt{\sigma^2 - 1}$$



$$\frac{d}{dx}\begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} = \epsilon \left[\frac{d\log(1-x)}{dx}\mathbb{M}_1 + \frac{d\log x}{dx}\mathbb{M}_2 + \frac{d\log(1+x)}{dx}\mathbb{M}_3 \right] \cdot \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix},$$

$$\mathbb{M}_1 = M_3 = \begin{pmatrix} -6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbb{M}_2 = \begin{pmatrix} 6 & 0 & -1 \\ 0 & -2 & -2 \\ 12 & 2 & 0 \end{pmatrix}$$

symbol alphabet:
harmonic polylogs
[Remiddi, Vermaseren' 99]

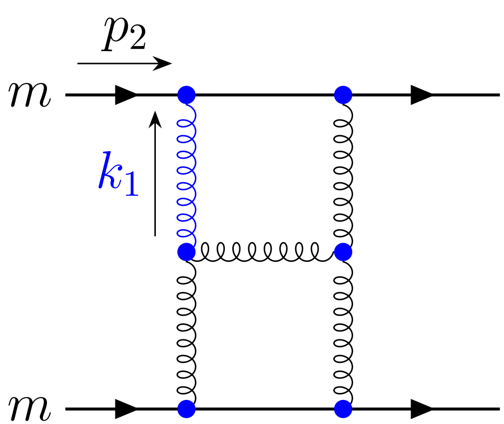
Physical input: PN expansion has no log(v) ~ log(1-x) singularity

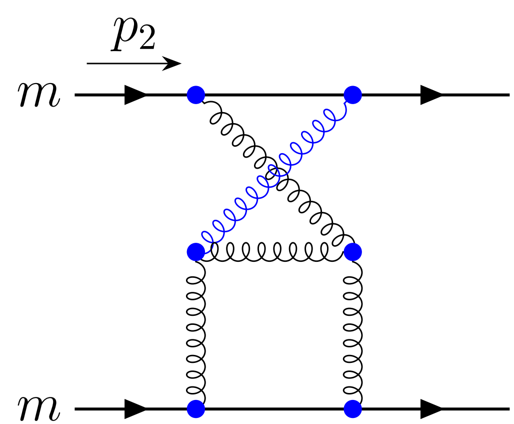
$$\mathbb{M}_1 \cdot \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} = 0 \implies I_1\big|_{x=1} = I_2\big|_{x=1} = 0, \quad I_3\big|_{x=1} = Ct^{-2\epsilon}/\epsilon^2$$

$$I_1 = -(Ct^{-2\epsilon}/\epsilon)\log x + \mathcal{O}(\epsilon^0) \sim -4(\log t)\,\mathrm{arcsinh}\sqrt{\frac{\sigma-1}{2}}.$$

COMPARISON WITH UNEXPANDED INTEGRAL

$m_1 = m_2$ results in [Bianchi, Leoni, 1612.05609], thanks to *Loopedia.org*.





$$s = (p_1 + p_2)^2 = -\frac{(1-x)^2}{x}$$

25 master integrals

$$(I_H)\big|_{\epsilon^0} = -\frac{4}{3}\log(-t)\frac{x}{1-x^2}\left(\pi^2 \log x + \log^3 x\right) + (\text{non-singular in } t)$$

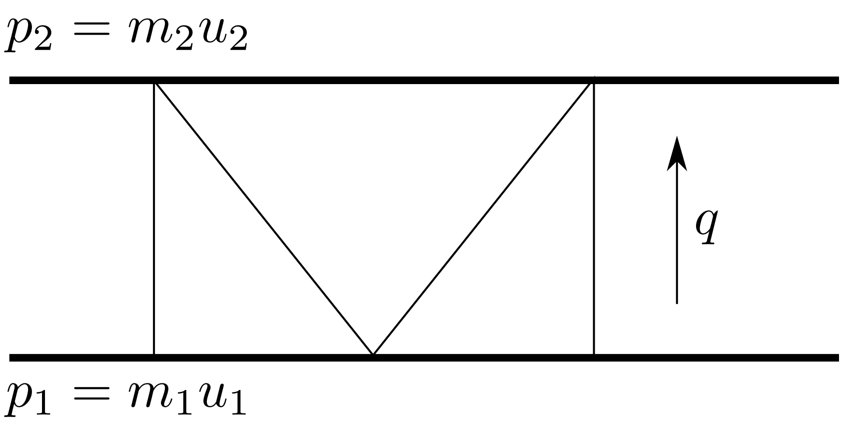
$$(I_{xH})\big|_{\epsilon^0} = -\frac{4}{3}\log(-t)\frac{-x}{1-x^2}\left(\pi^2 \log(-x) + \log^3(-x)\right) + (\text{non-singular in } t)\,.$$

$$\log(x) \to \log(-x) + i\pi, \quad (I_H + I_{xH})\big|_{\epsilon^0} = \underbrace{4\pi^2 \log(-t)\log(-x)}_{} + \text{imaginary}$$

Agrees with our solutions of cut DEs in small t limit.

FIRST GLIMPSE: 4PM / 3 LOOPS *(Preliminary)*

[Ita, Ruf, MZ, in progress]



$$u_1 \cdot u_2 = \sigma = \frac{1+x^2}{2x}, \quad 1 > x > 0$$

- Simplest 3-loop topology; can work on *max. cut*.
  Obtain 4×4 system of DEs. Symbol alphabet: $x, 1 \pm x, 1 \pm ix$.
- Regularity at $x = 1$ fixes boundary conditions up to overall factor.
- ***Scalar integral*** $\propto \frac{\sqrt{-t}}{\sigma}\Big\{1 + \epsilon\big[14(x^2-1)^2 - 32x^2 \log\sigma + \ldots\big]\Big\}$.
- Also solved 2- & 3-loop DEs ***without cuts***. Matches FIESTA 4 [Smirnov]
  Status of other topologies: need to clean up ε-dependent denominators for running `epsilon`.

FUTURE OUTLOOK

- Higher orders within reach. Double copy + EFT methods expected to scale well.

- Relativistic integration to fully settle questions about velocity resummation; demonstrated feasibility at 3 loops.

- Scattering amplitudes begin to impact gravitational astronomy. Rich physics opportunities:

  Spin, finite-size effects in PM expansion [Bini, Damour, '17; Vines, '17, Bini, Damour, '18; Guevera, Ochirov, Vines, '18; Vines, Steinhoff, Buonanno, '18; Chung, Huang, Kim, Lee, '18; Maybee O'Connell, Vines, '19; Guevara, Ochirov, Vines, '19]

  Tail effect / nonlocal potential from ultrasoft modes
  [Porto, Rothstein, '17]