Equivalence of the Resonating-Valence-Bond and Fractional Quantum Hall States[1]

Literature Review

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Soluyanov Group Meeting, University of Zurich

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[1] V. Kalmeyer & R. B. Laughlin, Phys. Rev. Lett. **59**, 2095 (1987)

Contents

Introduction

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Key concepts to cover:

1. Anderson's resonating valence bond state[2]
2. Holstein-Primakoff transformation[3]
3. Laughlin wavefunction[4]

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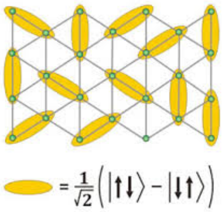
[2] P. W. Anderson, Mater. Res. Bull. **8**, 153 (1973)
[3] T. Holstein & H. Primakoff, Phys. Rev. **58**, 1098 (1940)
[4] R. B. Laughlin, Phys. Rev. Lett. **50**, 1395 (1983)

What is a resonating-valence-bond (RVB) state?

Consider a lattice of atoms with a spin d.o.f. at each site.

RVB state = sum over all arrangements of a tensor product of directed dimer states. It is a highly entangled quantum state for the spin degrees of freedom ('**quantum spin liquid**').



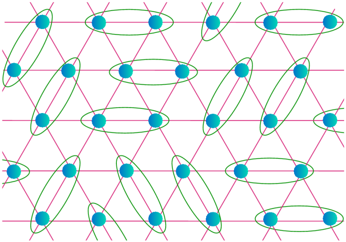
- △ Nearest-neighbor atoms interact via their outermost electrons. Hence the singlets are **valence-bonded**.
- △ Since all interactions are short-range (NN) & no preference for specific valence bond, the state is **liquid-like**.
- △ The symmetric superposition of all possible dimer coverings means that the system can **resonate** between configurations.

Model

Consider frustrated spins with an antiferromagnetic ($J > 0$) interaction on a 2D triangular lattice.

$$H_{\mathrm{AF}} = J \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j$$

This is the Heisenberg Hamiltonian for nearest neighbor interactions, with $\mathbf{S}_j = \boldsymbol{\sigma}_j/2$.



The ground state of this system is postulated to not be a Néel 'solid' of spins fixed at sites, but rather a mobile 'liquid' of dimers.

Holstein-Primakoff transformation I

Aim to show that Anderson's Hamiltonian for the RVB ground state is *equivalent* to the FQH Hamiltonian for bosons on a lattice.

$$H_{\mathrm{AF}} = J \sum_{\langle ij \rangle} \left[S_{z,i} S_{z,j} + \frac{1}{2} \left(S_i^+ S_j^- + S_i^- S_j^+ \right) \right]$$

$$S^+ = \sqrt{2s}\sqrt{1 - \frac{a^\dagger a}{2s}}\, a \approx \sqrt{2s}\, a$$

$$S^- = \sqrt{2s}\, a^\dagger \sqrt{1 - \frac{a^\dagger a}{2s}} \approx \sqrt{2s}\, a^\dagger$$

$$S_z = (s - a^\dagger a)$$

The **Holstein-Primakoff transformation** is a (non-linear) map from a problem of coupled spins to a problem of coupled oscillators.

NB: It can be shown that $[a_i, a_j^\dagger] = \delta_{i,j}$ recovers the spin commutation relations.

Holstein-Primakoff transformation II

Quick interpretation:

- △ $S^{\pm}$ and $a, a^{\dagger}$ both shift up and down a ladder of states.
- △ This is why S_z and the oscillator number operator are simply related ($S_z = s - n_b$).
- △ However, in the spin case the ladder is *finite* ($\mathcal{H}_{\text{spin}} = \{|s\rangle, \dots, |-s\rangle\}$), whereas in the oscillator case the ladder is *semi-infinite* ($\mathcal{H}_{\text{osc}} = \{|s\rangle, \dots, |-s\rangle, |-s-1\rangle, \dots, |-\infty\rangle\}$).
- △ Hence, $S^{+} \propto a$ would be too naive – we need something to stop us lowering below $S_z = -s$.
- △ For this, we have the factors under the square roots in $S^{\pm}$ to enforce that $n_{\text{b}} \leq 2s \Rightarrow \dim(\mathcal{H}_{\text{spin}}) = \dim(\mathcal{H}_{\text{osc}})$ ✓
- △ Having said this, Holstein-Primakoff is most often linearized to yield a quadratic free Hamiltonian[5].

[5] see refs on "$1/s$-expansion" and "linear spin wave theory" for details

Holstein-Primakoff transformation III

We now turn our Heisenberg Hamiltonian into a quadratic oscillator Hamiltonian.

Holstein-Primakoff transformation for H_{AF} gives

$$H_{\mathrm{AF}} = \underbrace{sJ \sum_{\langle ij \rangle} \left(a_j^\dagger a_i + a_i^\dagger a_j \right)}_{\hat{T}} + \underbrace{2sJ \sum_{\langle ij \rangle} a_j^\dagger a_i^\dagger a_j a_i - 12sJ \sum_i a_i^\dagger a_i}_{\hat{V}},$$

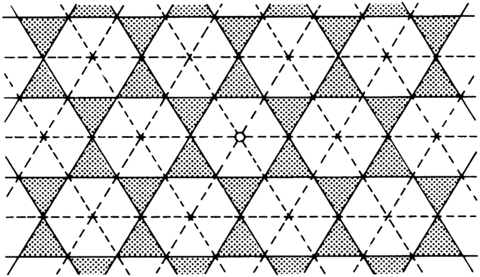
up to a constant, where $s = 1/2$.

Since we have mapped the spin problem to a boson for every spin-$\uparrow$, we finally enforce that we only allow one boson per site (hardcore constraint) to complete the transformation.

Gauge transformation I

Q: The kinetic energy term in our Hamiltonian does not have the *free-particle form* due to the strictly positive pre-factor ($J > 0$). How can we change this to the free-particle form?

A: Perform a gauge transformation, such that some of the bonds are negative and some positive. e.g. $\tilde{J}_{ij} = \pm J$ as shown in the figure below.



$+$ bond $=$ solid line

$-$ bond $=$ dashed line

Gauge transformation II

- △ After gauge transformation, this system corresponds to bosons of charge e^* hopping on the lattice in a perpendicular magnetic field!
- △ Bosons hop on the triangular lattice with amplitude J. In a perpendicular magnetic field, $J \to Je^{\mathrm{i}\theta_{ij}}$, where $\theta_{ij} = (2\pi/\phi_0)\int_i^j \mathbf{A} \cdot \mathrm{d}\mathbf{l}$ are the Peierl's phases, $\mathbf{A}$ is the vector potential, and $\phi_0 = hc/e^*$ is the flux quantum.
- △ We fix the magnitude of the fictitious B-field such that we have one magnetic flux per unit cell.

$$\text{area of unit cell} = \text{irreducible magnetic area}$$

$$\frac{\sqrt{3}a^2}{2} = 2\pi l_0^2$$

where a is the lat. const. and $l_0 = \sqrt{\frac{\hbar c}{e^* B}}$ is the mag. length[6].

[6] see, for example, section on Landau quantization in Ezawa's book

Example of Peierl's substitution I

In order to achieve a magnetic field $\mathbf{B} = \nabla \times \mathbf{A} = B\hat{\mathbf{z}}$, we choose to work in symmetric gauge $\mathbf{A} = \frac{B}{2}(x\hat{\mathbf{y}} - y\hat{\mathbf{x}})$.

By taking an appropriate parameterization, for example

$$x = X_i + (X_j - X_i)\tau \quad \text{and} \quad y = Y_i + (Y_j - Y_i)\tau$$

where $\tau \in [0, 1)$, we find that the Peierl's phases can be written as

$$\theta_{ij} = \frac{\pi B}{\phi_0}\left[X_i \underbrace{(Y_j - Y_i)}_{\Delta Y} - Y_i \underbrace{(X_j - X_i)}_{\Delta X}\right].$$

Consider a point with rectangular coordinates (mb, nc) in the (x, y) basis, where $b = a/2$, $c = \sqrt{3}a/2$, and m, n are integers.

Example of Peierl's substitution II

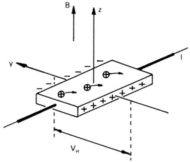
Consider $(X_i, Y_i) = (mb, nc)$ with $b = a/2$ and $c = \sqrt{3}a/2$, with

$$\theta_{ij} = \frac{\pi B}{\phi_0}\left[X_i \Delta Y - Y_i \Delta X\right].$$

Ex. 1 Hopping to right implies $\Delta Y = 0$, $\Delta X = a$, $Y_i = nc$, and hence then Peierl's phase yields $\theta_{ij} = (\pi B/\phi_0)(-nca) = -\pi n_\phi n$, where n_ϕ is the flux density per unit cell. Since $n_\phi = 1$, hopping to the right will give $e^{\mathrm{i}\theta_{ij}} = \pm 1$, depending on n.

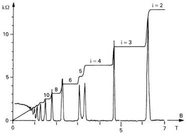
Ex. 2 Hopping to the upper-right implies $X_i = mb$, $\Delta Y = c$, $Y_i = nc$, and $\Delta X = b$. Hence the Peierl's phase yields $\theta_{ij} = (\pi B/\phi_0)(bc(m-n)) = -\frac{\pi}{2} n_\phi (m-n)$. Since $n_\phi = 1$ and $(m-n)$ is never odd, hopping will again yield $e^{\mathrm{i}\theta_{ij}} = \pm 1$ depending on the coordinates of the site.

What is a fractional quantum Hall (FQH) state?



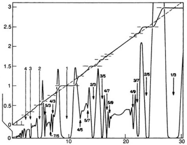
Classical Hall Effect (Hall, 1879)

- explained using Drude model
- sample impurity irrelevant
- classical regime



Integer **Q**uantum **H**all **E**ffect (Klitzing et al., 1980)

- fully-filled Landau levels
- sufficiently disordered sample ($0 \ll V_{\text{disorder}} \ll \hbar\omega_c$)
- quantum regime ($k_B T \ll \hbar\omega_c$)



Fractional **Q**uantum **H**all **E**ffect (Tsui et al., 1982)

- partially-filled Landau levels
- sufficiently pure sample ($V_{\text{disorder}} \ll V_{\text{Coulomb}} \ll \hbar\omega_c$)
- extreme quantum regime ($k_B T \lll \hbar\omega_c$)

FQH state $\rightarrow$ Hall conductivity $\sigma_{xy} = \nu \frac{e^2}{h}$, where ν is a rational fraction.

Laughlin wavefunction I

Q: The FQHE is an interaction-dominated many-body problem with a prohibitively large Hilbert space. How do we proceed to obtain the ground-state wavefunction?

Laughlin wavefunction I

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Laughlin wavefunction II

Consider a free particle of charge e^* and mass m moving in a perpendicular magnetic field.

$$H = -\frac{1}{2m}(\nabla - \mathrm{i}e^*\mathbf{A})^2$$

- symmetric gauge $\mathbf{A} = \frac{B}{2}(x\hat{\mathbf{y}} - y\hat{\mathbf{x}})$
- complex coordinates $z = x + \mathrm{i}y$, $\partial_z = \frac{1}{2}(\partial_x + \mathrm{i}\partial_y)$
- cyclotron frequency $\omega_c = e^*B/m$

$$H = -\frac{2}{m}\left(\partial_z - \frac{e^*B\bar{z}}{4}\right)\left(\partial_{\bar{z}} + \frac{e^*Bz}{4}\right) + \frac{\omega_c}{2}$$

We therefore know that states for which $(\partial_{\bar{z}} + \frac{qBz}{4})\psi = 0$ are in the **lowest Landau level (LLL)**. These states are of the form:

$$\psi(z,\bar{z}) = f(z)\exp\left(-\frac{e^*B}{4}|z|^2\right).$$

Laughlin wavefunction III

$$\psi(z,\bar{z}) = f(z)\exp\left(-\frac{e^*B}{4}|z|^2\right),$$

where $f(z)$ is an arbitrary analytic function (i.e. power series). Landau levels are *highly degenerate* $\Rightarrow$ freedom to choose the coefficients in the power series.

Let us now fill the LLL with many non-interacting fermions by introducing a symmetric potential $V_{\text{harm}} = \frac{v}{2}|z|^2$ to lift the degeneracy. Since the potential acting on the basis states simply counts their degree, $V_{\text{harm}} f_n = v(1+n)f_n$, the ground state must fill the basis states from the bottom[7]. Hence, in this case

$$\Psi(z_1,\ldots,z_N) = \underbrace{\prod_{j<k}^{N}(z_j - z_k)}_{\text{Vandermonde determinant}}\exp\left(-\frac{1}{4l_0^2}\sum_{j=1}^{N}|z_j|^2\right)$$

[7] see e.g. Jain's book for more details

Laughlin wavefunction IV

Finally, Laughlin's ansatz is to generalise this ground-state wavefunction to the case of *fractionally*-filled Landau levels with **filling factor**

$$\nu \equiv \frac{N}{N_s} \equiv \frac{n}{n_\phi} = \frac{1}{m}, \quad \text{where} \quad \begin{cases} m \text{ is odd for fermions} \\ m \text{ is even for bosons} \end{cases}$$

$$\Psi_m(z_1, \ldots, z_N) = \underbrace{\prod_{j<k}^{N} (z_j - z_k)^m}_{\text{Jastrow factor}} \exp\left(-\frac{1}{4l_0^2} \sum_{j=1}^{N} |z_j|^2 \right)$$

- (anti)symmetric under particle exchange
- eigenstate of total angular momentum
- Coulomb repulsion is included via two-body correlations

Laughlin wavefunction V

Laughlin wavefunction is *fundamentally different* to the fully-filled product state we wrote down before. It's also not the actual ground state for FQH. But it has the same **fractional statistics** and **topological order** as the actual FQH ground state.

excitations $\Rightarrow$ "new state of matter" [Laughlin, 1983]

Here, let us consider a 2DEG with N_b interacting bosons in a perpendicular magnetic field. We take Laughlin's ansatz for the solution at 1/2-filling (i.e. $\# \uparrow$-spins $= \# \downarrow$-spins):

$$\Psi_2(z_1, \ldots, z_N) = \prod_{j<k}^{N} (z_j - z_k)^2 \exp\left(-\frac{1}{4l_0^2} \sum_{j=1}^{N} |z_j|^2 \right)$$

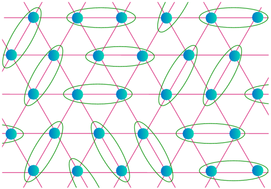
Q: How well does this describe the spin model with puported RVB ground state?

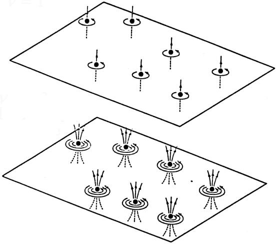
Are these two states equivalent?

More generally, how do we quantify whether two states are equivalent?

1. wavefunction overlap / agreement of ground state energies
2. adiabatic continuity between the two Hamiltonians

$$H_{\mathrm{AF}} \overset{?}{\leftrightarrow} H_{\mathrm{FQH}}$$





Wavefunction overlap

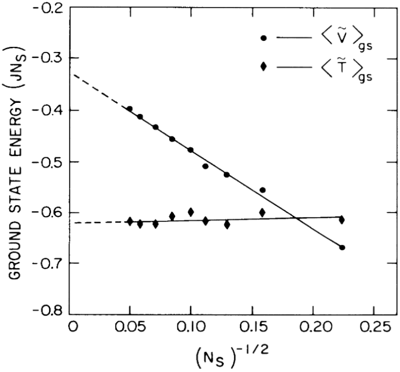


Figure: Ground-state energy of the $\nu = 1/2$ FQH system evaluated numerically using variational Monte Carlo.

$$E_{\mathrm{VMC}} = \frac{\langle \Psi | H | \Psi \rangle}{\langle \Psi | \Psi \rangle} = \frac{\int |\Psi(\mathbf{r})|^2 E_{\mathrm{L}}(\mathbf{r}) \mathrm{d}\mathbf{r}}{\int |\Psi(\mathbf{r})|^2 \mathrm{d}\mathbf{r}}$$

E_{RVB}	E_{FQH}
-0.95 ± 0.02	-0.94 ± 0.02

The ground state energies, obtained from numerical finite-size extrapolations, agree within 2% ✓

Adiabatic continuity

General idea: write the two Hamiltonian in the same basis so that we can adiabatically tune from one to the other.

$$H(\kappa) = \kappa H_{\mathsf{AFM}} + (1-\kappa) H_{\mathsf{FQH}}, \quad \kappa \in [0,1)$$

Suggested basis: single boson basis orbitals

$$\phi_\alpha = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{4}(z - z_\alpha)^2\right)$$

Then the overlaps $S_{\alpha\beta} = \int \phi^*_\alpha(z)\phi_\beta(z)\mathrm{d}^2 z \to 0$ to tune to H_{AF}.

- △ Since the states are expected to be equivalent, this would support the idea that the RVB ground state is gapped, so that the *gap is stable* under adiabatic transformation.

1987 → present day

Q: Why does this matter?

A: Connection between doped quantum spin liquid Mott insulator and high-T_c superconductivity[8]. Can the FQHE provide further insight?

Progress in theory:

- △ not all quantum spin liquids are alike, and great efforts have been made to classify them[9]
- △ RVB states have now been shown to be the ground states of many model Hamiltonians[10]

Progress in experiment:

- △ frustrated magnets in 2D and 3D, e.g. herbertsmithite, rare-earth dichalcogenides, triangular organics, etc.[11]

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[8] review by Lee et al., Rev. Mod. Phys. **78**, 17 (2006)

[9] e.g. A. M. Essin & M. Hermele, Phys. Rev. B **87**, 104406 (2013)

[10] e.g. R. Moessner & S. L. Sondhi, Phys. Rev. Lett. **86**, 1881 (2001)

[11] review by L. Balents, Nature **464**, 199–208 (2010)

Conclusion

- △ The AF Heisenberg model can be transformed to a Hubbard model in the low energy sector
- △ The resultant Hubbard model is equivalent to bosons hopping in a perpendicular magnetic field
- △ The ground states and physics of the spin and FQH systems are analogous

Potential research directions:

- ? Is it possible to adiabatically transform between the Hamiltonians?
- ? Is the gap robust to such a transformation / such transformations in general?
- ? To what extent is FQH physics reflected in quantum spin liquids and high-T_c superconductivity?

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Thank you for listening!