General Relativity
Sheet 4

HS 2019
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Exercise 1 [Motion of a particle on a sphere]

A particle sits on a sphere. The equations of motion can be derived from the action

$$S = \int_A^B ds = \int_A^B \sqrt{g_{ij} dx^i dx^j}, \tag{1}$$

where g_{ij} is the metric of a sphere with radius $r = a$.

(i) Find the metric of the sphere.

(ii) Use (1) to show that the trajectory the particle follows is that of a great circle.

Exercise 2 [Global Positioning System / Clocks in Space]

The Global Positioning System (GPS) consists of at least 24 satellites that are orbiting the Earth at a distance of $h = 26'600$ km from its center with a velocity of $v \sim 3.9$ km/s. All the satellites carry atomic clocks which are synchronized such that they all show the same (GPS) time. At certain time intervals, the satellites simultaneously emit a signal which carries their orbital data and the time t_e when the signal was emitted. From the time when the signal was emitted, a GPS receiver can compute the position of the satellite through the orbital data received and thus ends up having the information $(t_e, \vec{x}_i)$ about satellite i, where $\vec{x}_i$ is its position in a fixed coordinate frame. This can be used to compute the position $\vec{x}_0$ of the receiver.

a) Due to relativistic effects, the satellite clocks will run with a different speed than clocks on Earth. In the lecture you have derived the Newtonian limit for general relativistic effects. Assuming that $v^2, \phi \ll c^2$, we find that $g_{00} = 1 + \frac{2\phi}{c^2}, g_{ij} = -\delta_{ij}$, up to $\mathcal{O}\left(\frac{1}{c^3}\right)$. Thinking as an observer far away from Earth (neglecting Earth's motion), compute the relation between infinitesimal elements $\mathrm{d}\tau_E$ of Earth coordinate time and $\mathrm{d}\tau_S$ of satellite coordinate time.

   Hint: Only speed affects clocks, but not acceleration per se!

b) Expand the relation you found in a) up to $\mathcal{O}\left(\frac{1}{c^3}\right)$, assuming that $v^2, \phi \ll c^2$. How much does a satellite clock run faster/slower than a clock on Earth? What is the absolute error after one day? Compare SR and GR effects.

c) One can get rid of (leading order) relativistic effects by changing the factory time speed of the atomic clock. A Cesium clock runs at a frequency of 10.23 MHz. How has this frequency to be changed before the launch of a satellite such that it runs still at that frequency once in orbit?

d) Compute the travel time for a photon travelling the distance between the satellite and the Earth. Why is the magnitude of this effect neglegible compared to the relativistic corrections to the satellite clocks discussed in b) ?

Hint: Light satisfies $\mathrm{d}s^2 = g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} = 0$, where λ is an affine parameter of the photon trajectory. Use the metric from a) in spherical coordinates to compute the travel time $t_{S\to E}$ from satellite to Earth.

e) Assuming that the GPS receiver has an atomic clock synchronized with Earth coordinate time, what would the error in distance determination to just one satellite be after it has been in orbit with no adjusted clock for one day? Why does this not matter for GPS navigation in the end?

Exercise 3 [Motion of a test particle in an arbitrary metric]

A particle finds itself on a manifold endowed with an arbitrary metric $g_{\alpha\beta}$, which depends on the 4-coordinates.

$$S = \int_A^B ds = \int_A^B \sqrt{g_{\alpha\beta}dx^\alpha dx^\beta} \tag{2}$$

Using the Euler-Lagrange equations, show that the trajectory of the particle, parametrized by λ, is given by solutions to

$$\frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\delta\beta}\frac{dx^\delta}{d\lambda}\frac{dx^\beta}{d\lambda} = 0, \tag{3}$$

where Γ are the Christoffel Symbols, given by

$$\Gamma^\alpha_{\delta\beta} = \frac{1}{2}g^{\alpha\gamma}\left[\frac{\partial g_{\gamma\delta}}{\partial x^\beta} + \frac{\partial g_{\gamma\beta}}{\partial x^\delta} - \frac{\partial g_{\delta\beta}}{\partial x^\gamma}\right]. \tag{4}$$