Exercise 1. ***Magnetic monopoles***

Consider the Lagrangian density

$$\mathscr{L} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu,a} + \frac{1}{2}D_\mu\phi D^\mu\phi - \frac{\lambda}{8}\left(\phi^2 - \eta^2\right)^2 , \tag{1}$$

where ϕ is a scalar field in the three-dimensional representation of $SO(3)$, the covariant derivative is given by

$$D_\mu\phi_a = \partial_\mu\phi_a - g\epsilon_{abc}A^b_\mu\phi_c \tag{2}$$

and the field strength tensor is

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g\epsilon_{abc}A^b_\mu A^c_\nu . \tag{3}$$

The magnetic monopole solution can be parametrised by the Ansatz

$$\phi_a = \frac{H(\xi)}{\xi}\eta\frac{x_a}{r} , \qquad A^a_i = \frac{\epsilon_{aij}x_j}{gr^2}\left(K(\xi) - 1\right) , \tag{4}$$

where $\xi = g\eta r$.

1. Show that the covariant derivative is given in terms of this Ansatz by

$$D_i\phi_a = \frac{K(\xi)H(\xi)}{gr^4}\left(r^2\delta_{ai} - x_a x_i\right) + \left(\xi H'(\xi) - H(\xi)\right)\frac{x_a x_i}{gr^4} . \tag{5}$$

2. Show that the energy of the magnetic monopole is given by

$$E = \frac{4\pi\eta}{g}\int_0^\infty \mathrm{d}\xi \frac{1}{\xi^2}\left[\frac{1}{2}(\xi H' - H)^2 + H^2K^2 + (\xi K')^2 + \frac{1}{2}(K^2 - 1)^2 + \frac{\lambda}{8g^2}(H^2 - \xi^2)^2\right] . \tag{6}$$

3. Show that the energy is minimised for

$$\begin{aligned} \xi^2 K'' &= KH^2 + K(K^2 - 1) , \\ \xi^2 H'' &= 2K^2H + \frac{\lambda}{2g^2}H(H^2 - \xi^2) . \end{aligned} \tag{7}$$