uzh | eth | zürich

General Relativity
Sheet 10

HS 2019
Prof. P. Jetzer

M. Haney, S. Tiwari, A. Boitier, M. Ebersold, A. Dei, L. Fritz
Issued: 20.11.2019
Course website: `http://www.physik.uzh.ch/en/teaching/PHY511`
Discussion: 28.11.2019

Exercise 1 Schwarzschild geometry

Consider the Schwarzschild geometry given by

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - \frac{1}{1 - \frac{2GM}{c^2 r}} dr^2 - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right).$$

A photon comes from infinity with impact parameter b.

a) Show that if $b^2 < 27\left(GM/c^2\right)^2$, the photon crosses the Schwarzschild radius.

b) Show that for b^2 suitably close to $27\left(GM/c^2\right)^2$, the photon can be made to orbit an arbitrary number of times before escaping to infinity.

Exercise 2 Radial fall into a black hole

Consider a particle which is released from $r = R$ and falls radially inward in a Schwarzschild spacetime.

Show that the motion can be parametrized cycloidally:

$$r = \frac{R}{2}(1 + \cos\eta)$$
$$\tau = \sqrt{\frac{R^3}{8GM}}(\eta + \sin\eta),$$

with $\eta \in [0, \pi]$. From this result, deduce that the proper time elapsed when the particle reaches the Schwarzschild radius $r_S = 2GM/c^2$ is finite, as well as the proper time elapsed when the particle reaches the singularity at $r = 0$.

Exercise 3 Inside a black hole

Consider a particle (not necessarily on a geodesic) that has fallen inside the event horizon of a black hole, $r < r_S = 2GM/c^2$. Show that the radial coordinate must decrease at a minimum rate given by

$$\left|\frac{dr}{d\tau}\right| \geqslant \sqrt{\frac{2GM}{r} - c^2}.$$

Calculate the maximum lifetime for a particle along a trajectory from $r = 2GM/c^2$ to $r = 0$. Express this in seconds for the supermassive black hole Sagittarius A^* in the center of our galaxy, whose mass is about $8 \cdot 10^{36}$ kg. Show that this maximum proper time is achieved by a radial free fall.

Hint: Use $u_\mu u^\mu = c^2$ to get an inequality for u^r.