

**Exercise 1. Axial anomaly in massive QED**

Consider massive QED in 4 dimensions:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\not{D}\psi - m\bar{\psi}\psi. \quad (1)$$

Compute the divergence of the axial current, i.e.  $\partial_\mu J^{\mu 5}$ , with

$$J^{\mu 5} = \bar{\psi}\gamma^\mu\gamma^5\psi. \quad (2)$$

*Hint:* in order to get a non-vanishing result even for vanishing fermion mass, you have to define the axial current as

$$\text{symm} \lim_{\delta \rightarrow 0} \left( \bar{\psi} \left( x + \frac{\delta}{2} \right) \gamma^\mu \gamma^5 \exp \left[ -ie \int_{x-\delta/2}^{x+\delta/2} dz A(z) \right] \psi \left( x - \frac{\delta}{2} \right) \right), \quad (3)$$

where the symmetric limit means

$$\text{symm} \lim_{\delta \rightarrow 0} \left( \frac{\delta^\mu}{\delta^2} \right) = 0, \quad \text{symm} \lim_{\delta \rightarrow 0} \left( \frac{\delta^\mu \delta^\nu}{\delta^2} \right) = \frac{g^{\mu\nu}}{4}. \quad (4)$$

**Exercise 2. Fermion number non-conservation**

1. In massless QED, the divergence of the axial current is given by the Adler-Bell-Jackiw anomaly:

$$\partial_\mu J^{\mu 5} = -\frac{e^2}{16\pi^2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}. \quad (5)$$

Using this relation, show that:

$$\Delta N_R - \Delta N_L = \frac{e^2}{2\pi^2} \int d^4x \mathbf{E} \cdot \mathbf{B}, \quad (6)$$

where  $N_R$  and  $N_L$  are the numbers of right- and left-handed fermions,  $\mathbf{E}$  and  $\mathbf{B}$  are the electric and magnetic fields respectively.

2. By assuming  $A_0 = 0$ , show that the Hamiltonian for massless fermions is

$$H = \int d^3x \left[ \psi_R^\dagger (-i\sigma \cdot \mathbf{D}) \psi_R - \psi_L^\dagger (-i\sigma \cdot \mathbf{D}) \psi_L \right], \quad (7)$$

where  $\mathbf{D} = \nabla - ie\mathbf{A}$ .

3. Consider a background field given by  $A^\mu = (0, 0, Bx^1, A)$ , where  $B$  is constant and  $A$  is constant in space and varying adiabatically in time. In order to diagonalise the Hamiltonian of part 2, one has to find a solution of the eigenvalue equation  $-i\sigma \cdot \mathbf{D}\psi_R = E\psi_R$ , which can be obtained by writing

$$\psi_R = \begin{pmatrix} \phi_1(x^1) \\ \phi_2(x^1) \end{pmatrix} e^{i(k_2x^2 + k_3x^3)}. \quad (8)$$

Derive the differential equations obeyed by  $\phi_1$  and  $\phi_2$ , show that they are equivalent to the equation of a simple harmonic oscillator and use this observation to find the single-particle spectrum of the Hamiltonian.

4. Now assume that the system is set up in a cubic box of length  $L$ , with periodic boundary conditions. Then the momenta will be quantised as

$$k_i = \frac{2\pi n_i}{L} \quad n_i \in \mathbb{Z}. \quad (9)$$

Show that requiring the oscillation centre to stay inside the box leads to the condition

$$k_2 < eBL, \quad (10)$$

so that the degeneracy of each energy level is

$$\frac{eL^2 B}{2\pi}. \quad (11)$$

5. Finally, suppose to change the background  $A$  adiabatically by the amount

$$\Delta A^1 = \frac{2\pi}{eL}. \quad (12)$$

Show that the vacuum loses right-handed fermions. Similarly, compute the eigenvalues for the left-handed fermions and show that under the same change  $\Delta A^1$  the vacuum gains the same number of left-handed fermions. Show that these numbers agree with the global non-conservation law of part 1.

**Useful formulæ:** The chiral representation of the gamma matrices is given by

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (13)$$

with  $\sigma^\mu = (1, \sigma)$ ,  $\bar{\sigma}^\mu = (1, -\sigma)$  and  $\sigma^i$  are the Pauli matrices,

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (14)$$