Elementary Particle Theory – PHY452

Fall Semester 2020

Exercise Sheet 10

Exercise 1: Charged pion decay and helicity suppression

The charged pion π^+ is a bound state of quarks $\pi^+ = (u\bar{d})$ that can decay into a charged anti-lepton and a neutrino: $\pi^+ \to \ell^+\nu_\ell$, with $\ell = e,\ \mu$. Below the electroweak scale, this weak process can be described by the following semi-leptonic Fermi Lagrangian:

$$\mathcal{L} = \frac{G_F}{\sqrt{2}}(l_\alpha^\dagger h^\alpha + h_\alpha^\dagger l^\alpha) \tag{1}$$

where $l_\alpha = \bar{\psi}_\nu\gamma_\alpha(1-\gamma^5)\psi_\ell$ and $h_\alpha = \bar{\psi}_u\gamma_\alpha(1-\gamma^5)\psi_d$ are leptonic and hadronic currents with the typical $V-A$ structure describing weak interactions, and G_F is the Fermi constant. In this exercise we will show that π^+ decays predominantly to anti-muons rather than positrons.

a) Decompose the amplitude into a leptonic current and a hadronic matrix element. Now, use Lorentz symmetry to show that the hadronic matrix element can be written as

$$\langle 0|h^\alpha|\pi^+(q)\rangle = iq^\alpha f_\pi\,, \tag{2}$$

where q is the 4-momentum of the pion, and f_π is some Lorentz scalar. The quantity f_π is a decay constant that can only be determined with a non-perturbative computation. What is the mass dimension of f_π?

b) Use parity conservation to show that the vectorial part of the hadronic matrix element $\langle 0|\bar{\psi}_u\gamma^\alpha\psi_d|\pi^+(q)\rangle$ vanishes.
Hint: The pion is a pseudo-scalar particle.

c) Use (2) to show that the decay rate of the pion is

$$\Gamma(\pi^+ \to \ell^+\nu_\ell) \ = \ \frac{G_F^2}{8\pi}\, f_\pi^2\, m_\ell^2\, m_\pi \left(1 - \frac{m_\ell^2}{m_\pi^2}\right)^2 , \tag{3}$$

where $m_{\pi,\ell}$ are the masses of the pion and lepton. Show that the decay channel into positrons is $\sim 10^{-4}$ times more suppressed than the anti-muon channel. Compare this result with what you would obtain using naive dimensional analysis.

d) Which intermediate vector bosons of the electroweak theory is exchanged when the charged pion decays? Write down the Feynman diagrams.

e) Working in the rest frame of the π^+, use conservation of angular momentum during the decay to show that the emitted charged anti-lepton must have a left-hand helicity. What would be the helicity of the emitted lepton for a negatively charged pion $\pi^- = (\bar{u}d)$ decaying as $\pi^- \to \ell^-\bar{\nu}_\ell$? Discuss the limit of massless charged leptons.
Hints: The pion has spin 0 and the neutrino (anti-neutrino) is a massless left-handed (right-handed) chiral field. Also, recall that helicity and chirality are the same for massless fermions.

f) Charged pions are copiously produced when high-energy cosmic rays interact with nuclei of atoms in the upper atmosphere. Using the information above and the fact that the muon decays $\sim 100\%$ of the time into $\mu^- \to e^- \bar{\nu}_e \nu_\mu$, estimate the fractions of atmospheric neutrinos $(\nu_e + \bar{\nu}_e) : (\nu_\mu + \bar{\nu}_\mu) : (\nu_\tau + \bar{\nu}_\tau)$ coming from pions.

Exercise 2: Electroweak gauge bosons

The covariant derivative of the $SU(2)_L \times U(1)_Y$ electroweak theory is

$$D_\mu \;=\; \partial_\mu + i\, g_1 \frac{Y}{2} B_\mu \;-\; i\, g_2 T^a\, W^a_\mu \tag{4}$$

where $g_{1,2}$ are gauge couplings, B_μ and W^a_μ are the corresponding gauge fields, Y is the hypercharge and T^a are the $SU(2)$ generators. The physical gauge bosons are given by the linear combinations,

$$A_\mu = \frac{g_2 B_\mu - g_1 W^3_\mu}{\sqrt{g_1^2 + g_2^2}}\,, \quad Z_\mu = \frac{g_1 B_\mu + g_2 W^3_\mu}{\sqrt{g_1^2 + g_2^2}}\,, \quad W^\pm_\mu = \frac{1}{\sqrt{2}}(W^1_\mu \mp W^2_\mu)\,, \tag{5}$$

describing the photon, the Z boson and the W boson, respectively.

a) Derive the transformation properties of the physical gauge fields with respect to global phase transformations induced by the diagonal generator $Q = T^3 + Y/2$.

b) Rewrite the covariant derivative in terms of the physical gauge fields A_μ, $W^\pm_\mu$ and Z_μ.

c) By checking how the physical gauge fields interact with fermions, derive a relation between the electromagnetic coupling e and the gauge couplings $g_{1,2}$.

d) Derive the couplings of the Z boson to the chiral quark fields u_L, u_R, d_L, d_R, fields.

Hints: In the standard model, the up and down quarks u_L, d_L with left-handed chiralities are components of an $SU(2)_L$ doublet (typically denoted q_L), while the right-handed fields u_R and d_R are $SU(2)_L$ singlets:

$$q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \quad u_R, \quad d_R.$$