MMP II
Exercise Sheet 10

FS 19
Prof. Ph. Jetzer

A. Boitier, M. Haney, A. Primo, S. Tiwari
https://www.physik.uzh.ch/en/teaching/PHY322

Issued: 30.04.2019
Due: 7.5.2019

Exercise 1 [Symmetric group (8 points)]
S_n denotes the symmetric group, i.e. the group of all permutations of n elements.

a) Verify that

$$\rho(\pi) := \mathrm{sgn}(\pi) \ , \quad \pi \in S_n \ , \tag{1}$$

is a representation. Use this result to show that the set of even permutations,

$$A_n = \{\pi \in S_n \,|\, \mathrm{sgn}(\pi) = 1\} \ , \tag{2}$$

is an invariant subgroup of S_n.

b) Prove that the trivial representation and (1) are the only one-dimensional representations of S_n.

Hint: Let be ρ a one-dimensional representation of S_n. First, show that $\rho(t_{kl})$ for all transpositions t_{kl} (swapping elements k and l) is either 1 or -1. Use (and check) the identity $t_{12}t_{2k}t_{12} = t_{2k}t_{12}t_{2k}$ for an arbitrary $k = 1, 3, 4, ..., n$. Then, use the fact that the one-dimensional representations are *abelian*.

Exercise 2 [Generators of the Lorentz group (6 points)]
V_2 is the group generated by the space-time transformations

$$\begin{aligned} \mathbf{1} &: (t,x) \mapsto (t,x) \\ P &: (t,x) \mapsto (t,-x) \\ T &: (t,x) \mapsto (-t,x) \end{aligned}$$

a) Write down the group table of V_2.

b) Determine all the conjugacy classes of V_2. *Hint:* V_2 is abelian.

c) Determine all the subgroups of V_2. Which ones are invariant?

d) Write down explicitly the regular representation $\Gamma^{(\mathrm{reg})}(g)$ of V_2.

Exercise 3 [Representation of the Lorentz group (11 points)]

$O(1,1)$ is the subgroup of $A \in GL(2,\mathbb{R})$ satisfying $A^t g A = g$ for $g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Let $\mathbb{R}^{1,1} := (\mathbb{R}^2, (\cdot,\cdot)_{1,1})$ with the inner product $(x,y)_{1,1} := x_0 y_0 - x_1 y_1$. $O(1,1)$ acts on $\mathbb{R}^{1,1}$ through matrix multiplication.

a) Write down a matrix representation for V_2, acting on $\mathbb{R}^{1,1}$.

b) Write down the character table for this representation. Then, write down the whole character table of V_2 (that is, the character table for all irreducible representations of V_2).

   Hint: Remember the conjugacy classes from ex. 1.

c) Show that if the matrix A is a solution of $A^t g A = g$, then also the elements $M\,A$ with $M \in V_2$ are solutions.

d) Let us find an explicit representation of the Lorentz group now. Set $A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, plug it into the equation and show that $A = \begin{pmatrix} z & \pm\sqrt{z^2-1} \\ \pm\sqrt{z^2-1} & \pm z \end{pmatrix}$ with $z \in \mathbb{R}$ and $sgn(b) = sgn(c)\,sgn(d)$.

e) Write down $\tilde{A}$ (A without the $\pm$ signs) and show that $\{M\tilde{A} \,|\, M \in V_2\}$ spans all matrices $A \in O(1,1)$.

   Hint: You can always do the transformation $z \mapsto -z$ without loss of generality. Use the sign condition from d).

f) Set $z = cosh(\lambda)$ and write down $\tilde{A}$.