Elementary Particle Theory – PHY452

Fall Semester 2020

Exercise Sheet 2

Exercise 1: Commutation relations

Consider a free complex scalar field theory described by the Lagrangian

$$\mathcal{L} = (\partial_\mu \phi)^\dagger (\partial^\mu \phi) - m^2 \phi^\dagger \phi \,. \tag{1}$$

The quantum field $\phi(x)$ in the Dirac picture is defined as

$$\phi(x) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{1}{2\omega_{\vec{k}}} \left[a(\vec{k})\, e^{-ik\cdot x} \,+\, b^\dagger(\vec{k})\, e^{ik\cdot x} \right] , \tag{2}$$

where $\omega_{\vec{k}} := \sqrt{|\vec{k}| + m^2}$, and the creation and annihilation operators satisfy the commutation relations

$$[a(\vec{k}), a^\dagger(\vec{k}')] = [b(\vec{k}), b^\dagger(\vec{k}')] = (2\pi)^3\, 2\omega_{\vec{k}}\, \delta^{(3)}(\vec{k} - \vec{k}') \tag{3}$$

with all other commutators $[a(\vec{k}), a(\vec{k}')]$, $[b(\vec{k}), b(\vec{k}')]$, ..., vanishing.

a) Derive an expression for the conjugate momenta field $\pi(x) = \delta\mathcal{L}/\delta\dot{\phi}(x)$ as a function of creation and annihilation operators.

b) Show explicitly that the following equal-time commutation relations hold:

$$[\phi(x), \pi(y)]_{x_0 = y_0} = i\delta^{(3)}(\vec{x} - \vec{y}) \tag{4}$$

$$[\phi(x), \phi(y)]_{x_0 = y_0} = [\phi(x), \phi^\dagger(y)]_{x_0 = y_0} = [\pi(x), \pi(y)]_{x_0 = y_0} = 0 \tag{5}$$

c) Show that $[\phi(x), \phi^\dagger(y)] = D(x - y) - D(y - x)$ where

$$D(z) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{1}{2\omega_{\vec{k}}} e^{-ik\cdot z} \,. \tag{6}$$

Exercise 2: ϕ^3 model

Consider the Lagrangian for an interacting real scalar field theory,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{2}\phi^2 + \frac{\lambda}{3!}\phi^3 \,. \tag{7}$$

a) Derive the classical equation of motion for ϕ.

b) Consider the $2 \to 2$ scattering process $\phi\phi \to \phi\phi$, draw all Feynman Diagrams at tree level and 1-loop level.

c) Consider the $2 \to 3$ scattering process $\phi\phi \to \phi\phi\phi$, write all tree-level Feynman Diagrams.

Exercise 3: Feynman's Propagator

In order to derive the Feynman rules to compute S-matrix elements for scattering processes, we will need to deal with quantities known as n-point *correlation functions*, of the form $\langle 0|T\phi(x_1)\phi(x_2)\cdots\phi(x_n)|0\rangle$, where T is the time-ordered operator. For two complex scalar fields this product is defined as

$$T\phi(x)\phi^\dagger(y) := \phi(x)\phi^\dagger(y)\theta(x^0-y^0) + \phi^\dagger(y)\phi(x)\theta(y^0-x^0) = \begin{cases} \phi(x)\phi^\dagger(y) & ,\forall\ x^0 > y^0 \\ \phi^\dagger(y)\phi(x) & ,\forall\ x^0 < y^0 \end{cases}$$

where $\theta(\cdot)$ is the Heavy-side or step function. We define the *Feynman propagator* as the two-point correlation function

$$G_F(x-y) := \langle 0|T\phi(x)\phi^\dagger(y)|0\rangle\,.$$

a) Show that the correlator $D(z)$ defined in eq. (6) is a solution to the homogenous Klein-Gordon equation $(\Box_z + m^2)D(z) = 0$.

b) Demonstrate that the Feynman propagator can be written as

$$G_F(x-y) = G_R(x-y) + D(y-x)\,,$$

where G_R is the retarded propagator defined by $G_R(x-y) = \theta(x^0-y^0)\langle 0|[\phi(x),\phi^\dagger(y)]|0\rangle$.

c) Show that the Feynman propagator is a *Green function* of the Klein-Gordon operator, i.e. that it satisfies

$$(\Box_x + m^2)\,G_F(x-y) \;=\; -i\delta^{(4)}(x-y)\,.$$

d) By taking the Fourier transform show that the Green function $G(z)$ of the Klein Gordon equation $(\Box_z + m^2)G(z) = -i\delta^{(4)}(z)$ can be written as an integral in 4 dimensions:

$$G(z) = i\int \frac{\mathrm{d}^4k}{(2\pi)^4}\,\frac{e^{-ik\cdot z}}{k^2-m^2}\,. \tag{8}$$