MMP 2
Solutions 6

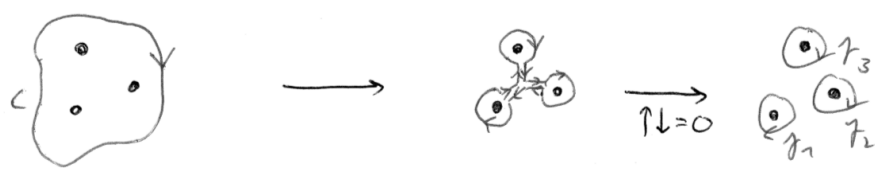
FS 19
Prof. Ph. Jetzer

Exercise 1

a) $f(z)$ holomorphic $\Rightarrow$ f can be expanded into a Laurent series around a with a certain radius of convergence:

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n = \underbrace{\sum_{n=-\infty}^{-1} c_n (z-a)^n}_{\substack{singular\ for\ z=a \\ \rightarrow\ singular\ part\ s(z)}} + \underbrace{\sum_{n=0}^{\infty} c_n (z-a)^n}_{\substack{=c_0\ for\ z=0 \\ \Rightarrow\ regular\ part\ r(z)}}$$

b) A path integral around a set of singular points can be contracted: Thus it is only valued by the



'residuals' of the enclosed singularities:

$$\int_C f(z)dz = \sum_j \int_{\gamma_j} f(z)dz \overset{f(z)=s(z)+r(z)}{=} \sum_j \left[\int_{\gamma_j} s(z)dz + \underbrace{\int_{\gamma_j} r(z)dz}_{=0} \right]$$
$$= \sum_j \int_{\gamma_j} s(z)dz$$

Using a):

$$\int_{\gamma_j} s(z)dz = \int_{\gamma_j} \sum_{n)-\infty}^{-1} c_n(z-a)^n = \sum_{n=1}^{\infty} c_{-n} \int_{\gamma_j} \frac{dz}{(z-a)^n}$$

$$\int_{\gamma_j} \frac{dz}{(z-a)^n} = \begin{cases} = 0 & \text{for } n > 1 \\ = 2\pi i & \text{for } n = 1 \end{cases}$$

$$\Rightarrow \int_{\gamma_j} s(z)dz = 2\pi i c_{-1}$$

$$\Rightarrow Res(f,a) \equiv \frac{1}{2\pi i} \int_{\gamma} f(z)dz = \frac{1}{2\pi i} \int_{\gamma} s(z)dz = c_{-1}$$

$$\Rightarrow \int_C f(z)dz = 2\pi i \sum_j Res(f, a_j)$$

a_j isolated singularities of f

c)

$$f(z) = \frac{\sin z}{z^2 - 2z} = \frac{\sin z}{z(z-2)}$$

$\Rightarrow$ poles in $z = 0$, $z = 2$ (single poles)

$$\Rightarrow Res(f, z_j) = \lim_{z \to z_j} (z - z_j) f(z) = \lim_{z \to z_j} \frac{(z - z_j)\sin z}{z(z-2)}$$

$$\Rightarrow Res(f, 0) = \lim_{z \to 0} \frac{z \cdot \sin z}{z(z-2)} = 0$$

($z = 0$ is not a real singularity since $\lim_{z \to 0} f(z) = -\frac{1}{2}$)

$$Res(f, 2) = \lim_{z \to 2} \frac{\sin z \cdot (z-2)}{z(z-2)} = \frac{\sin 2}{2}$$

d) i) Unit Circle

only $z = 0$ is inside the path, but we have seen that $Res(f, 0) = 0 \Rightarrow \oint_{\gamma_1} f(z)dz = 0$

ii) radius $R \to \infty$

$z = 0$ and $z = 2$ are inside the curve

$$\Rightarrow \lim_{R \to \infty} \oint_{\gamma_R} f(z)dz = 2\pi i [Res(f, 0) + Res(f, 2)]$$

$$= \pi i \sin 2$$

Exercise 2

a)

$$I = \int_0^{2\pi} \frac{1}{\cos^4(t) + \sin^4(t)} = \int_0^{2\pi} \frac{1}{\left(\frac{e^{it}+e^{-it}}{2}\right)^4 + \left(\frac{e^{it}-e^{-it}}{2i}\right)^4} \frac{ie^{it}}{ie^{it}} dt$$
$$\overset{z=e^{it}}{=} \frac{1}{i} \oint_\gamma \frac{1}{z} \frac{1}{\left(\frac{z+\frac{1}{z}}{2}\right)^4 + \left(\frac{zi\frac{1}{z}}{2i}\right)^4} dz$$

With γ the unit circle.

$$\Rightarrow f(z) = \frac{16}{z[(z+\frac{1}{z})^4 + (z-\frac{1}{z})^4]}$$
$$= \frac{16}{z[z^4 + 4\frac{z^3}{z} + 6\frac{z^2}{z^2} + 4\frac{z}{z^3} + z^{-4} + z^4 - 4\frac{z^3}{z} + 6\frac{z^2}{z^2} - 4\frac{z}{z^3} + z^{-4}]}$$
$$= \frac{8}{z(z^4 + 6 + z^{-4})} = \frac{8z^3}{z^8 + 6z^4 + 1} \equiv \frac{P(z)}{Q(z)}$$

$$Q(z): \ z^8 + 6z^4 + 1 = 0 \overset{w \equiv z^4}{\Rightarrow} w^2 + 6w + 1 = 0$$
$$\Rightarrow w_\pm = \frac{-6 \pm \sqrt{36-4}}{2} = -3 \pm \sqrt{8}$$
$$\Rightarrow z_{\pm k} = \sqrt[4]{w_\pm}, \ k = 1, 2, 3, 4$$

$|w_+| < 1 \Rightarrow |z_{+k}| < 1$ inside unit circle

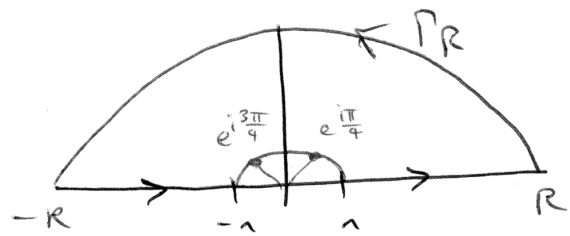
$|w_-| > 1 \Rightarrow |z_{-k}| > 1$ outside unit circle

$$\Rightarrow Res(f, z_{+k}) = \frac{P(z_{+k})}{Q'(z_{+k})} = \underbrace{\frac{8z_{+k}^3}{8z_{+k}^7 + 24z_{+k}^3}}_{no\ poles} = \frac{1}{z_{+k}^4 + 3}$$
$$\Rightarrow I = \frac{1}{i} 2\pi i \sum_{k=1}^{4} Res(f, z_{+k}) = \frac{4 \cdot 2\pi}{z_{+k}^4 + 3} = \frac{8\pi}{w_+ + 3} = \sqrt{8}\pi$$

b)

$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^4} dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{x^2}{1+x^4} \equiv I$$
$$f(x) = \frac{x^2}{1+x^4} \Rightarrow f(z) = \frac{z^2}{1+z^4}$$

with poles: $1 + z^4 = 0 \Rightarrow z_{1,2,3,4} = \left\{e^{i\frac{\pi}{4}}, e^{i\frac{3\pi}{4}}, e^{i\frac{5\pi}{4}}, e^{i\frac{7\pi}{4}}\right\}$



$$\oint_C f(z)dx = \int_{\Gamma_R} f(z)dz + I = 2\pi i \sum_j Res(f, z_j) \qquad (R \to \infty)$$

$$\lim_{R\to\infty} \int_{\Gamma_R} f(z)dz = \lim_{R\to\infty} \int_0^{2\pi} \frac{R^2 e^{2i\varphi}}{\underbrace{1 + R^4 e^{4i\varphi}}_{is\ larger\ for\ R\to\infty}} Rie^{i\varphi} d\varphi = 0$$

$$Res(f, z_j) \overset{single\ poles}{=} \lim_{z\to z_j} \frac{(z - z_j)z^2}{z^4 + 1} \equiv \lim_{z\to z_j} (z - z_j)\frac{P(z)}{Q(z)} = \lim_{z\to z_j} \frac{P(z)}{\frac{Q(z)-Q(z_j)}{z-z_j}}$$

$$= \lim_{z\to z_j} \frac{P(z)}{Q'(z_j)} = \frac{z_j^2}{4z_j^3} = \frac{z_j^3}{4\underbrace{z_j^4}_{=-1}} = -\frac{z_j^3}{4}$$

$$\Rightarrow I = 2\pi i \left[Res(f, e^{i\frac{i\pi}{4}}) + Res(f, e^{i\frac{i3\pi}{4}}) \right]$$

$$= -2\pi i \left[\frac{e^{i\frac{i3\pi}{4}}}{4} + \frac{e^{i\frac{i9\pi}{4}}}{4} \right] = \frac{-2\pi i}{4} \left[e^{i\frac{i3\pi}{4}} + \underbrace{e^{i\frac{i\pi}{4}}}_{2\pi\ periodic} \right]$$

$$= -\frac{2\pi i}{4} \left[\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} + \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} \right] = \pi \sin\frac{\pi}{4} = \frac{\pi}{\sqrt{2}}$$

c)

$$I = \int_0^\infty \frac{\sqrt{x}}{1 + x^3} dx = \lim_{R\to\infty} \int_0^R \frac{\sqrt{x}}{1 + x^3} dx$$

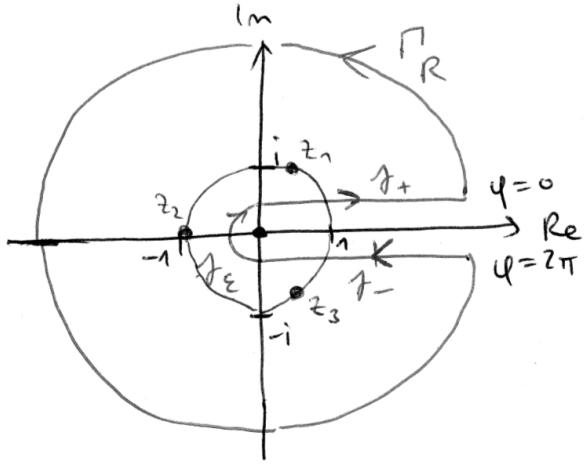
$$\Rightarrow f(z) = \frac{\sqrt{z}}{1 + z^3}$$

With poles $(1 + z^3) = 0 :\ e^{i\frac{\pi}{3}},\ e^{i\pi},\ e^{i\frac{5\pi}{3}}$

$\sqrt{z}$ is not a single-valued funtction!

$\to \sqrt{z} = \sqrt{re^{i\varphi}} = \sqrt{r}e^{i\frac{\varphi}{2}}$

$e^{i\frac{\varphi}{2}}$ is periodic, it needs to be constrained such that only two solutions $\to$ branch cut



$$\gamma_+ : z = x + i\varepsilon$$
$$\Rightarrow \sqrt{z} \overset{\varphi=0}{\to} \sqrt{x}e^{i\frac{0}{2}} = \sqrt{x}$$
$$\gamma_- : z = x - i\varepsilon$$
$$\Rightarrow \sqrt{z} \overset{\varphi=2\pi}{\to} \sqrt{x}e^{i\pi} = -\sqrt{x}$$

$$\oint\limits_C f(z)dz = \int\limits_{\Gamma_R} f(z)dz + \int\limits_{\gamma_\varepsilon} f(z)dz + \int\limits_{\gamma_+} f(z)dz + \int\limits_{\gamma_-} f(z)dz$$

$$\lim_{R\to\infty}\int\limits_{\Gamma_R} f(z)dz = \lim_{R\to\infty}\int\limits_0^{2\pi} \frac{\sqrt{Re^{i\varphi}}}{1+R^3e^{i3\varphi}} iRe^{i\varphi}d\varphi = \lim_{R\to\infty}\int\limits_0^{2\pi} \frac{R^{\frac{3}{2}}}{1+R^3e^{i3\varphi}} ie^{i\varphi}d\varphi = 0$$

$$\lim_{\varepsilon\to 0}\int\limits_{\gamma_\varepsilon} f(z)dz = \lim_{\varepsilon\to 0}\int\limits_{\frac{3\pi}{2}}^{\frac{\pi}{2}} \frac{\overbrace{\varepsilon^{\frac{3}{2}}}^{\to 0}}{1+\underbrace{\varepsilon^3e^{i3\varphi}}_{\to 0}} ie^{i\varphi}d\varphi = 0$$

$$\lim_{\varepsilon\to 0}\int\limits_{\gamma_+} f(z)dz = \int\limits_0^R \frac{\sqrt{x}}{1+x^3}dx$$

$$\lim_{\varepsilon\to 0}\int\limits_{\gamma_-} f(z)dz = \int\limits_R^0 \frac{-\sqrt{x}}{1+x^3}dx = \int\limits_0^R \frac{\sqrt{x}}{1+x^3}dx$$

$$\Rightarrow \oint\limits_C f(z)dz = 2\cdot I = 2\pi i\sum_j Res(f, z_j)$$

$$
\begin{aligned}
Res(f, z_j) &= \lim_{z \to z_j} (z - z_j) f(z) = \lim_{z \to z_j} (z - z_j) \frac{\sqrt{z}}{1 + z^3} \\
\Rightarrow I &= \pi i \left[\frac{1}{3 \underbrace{e^{i\frac{\pi}{2}}}_{=i}} + \frac{1}{3 \underbrace{e^{i\frac{3\pi}{2}}}_{=-i}} + \frac{1}{3 \underbrace{e^{i\frac{5\pi}{2}}}_{=i}} \right] \\
&= \frac{\pi}{3} [1 - 1 + 1] = \frac{\pi}{3}
\end{aligned}
$$

Exercise 3

The only singularity is hidden in $\ln(z)$, i.e. in $z = 0$. Apparently the curve does not enclose $z = 0$, so

$$
\oint_C \sin^2 \left(z^4 - 43z + 26^z \right) \exp \left(z^2 + 8 \cos \left(\ln(z) \right) \right) dz = 0
$$