The *deepest* cut: Peaks and Cusps

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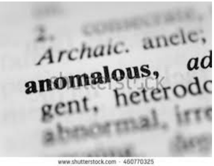
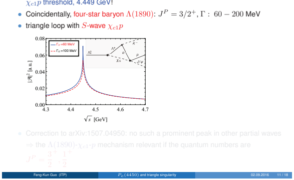
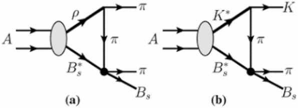


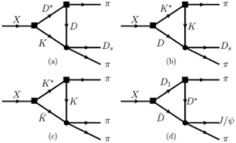
Zurich, 4 December 2018

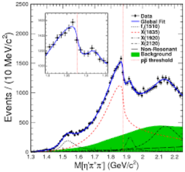


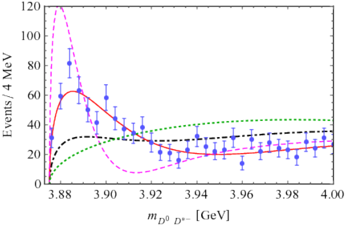
Why worry about cusps?[1]



***X*(5568)** ?







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[1] Resonances are normally observed as peaks in certain invariant mass distributions; however, a question arises: is a peak necessarily due to the presence of a resonance? Are there peaks produced by kinematic singularities?

Landau equations [2] for a given Feynman integral are a set of kinematic constraints that are necessary for the appearance of a pole or branch point in the integrated function (as a function of external kinematics and masses). Landau equations admit many families of solutions which are naturally classified as leading Landau singularities (LLS), sub-leading Landau singularities (SLLS), sub-sub-leading (S^2LLS) etc.

A triangle singularity is a logarithmic branch point, which would produce an infinite reaction rate if it appears in the physical region. The finite width, introduced by the complex-mass scheme, moves the singularity into the complex plane, and the differential reaction rate can have a finite peak due to the proximity of the singularity.

[2] Usually Landau-Nakanishi equations are called Landau equations for short, see also Bjorken (thesis, Stanford Univ. 1959)

let L be the number of internal lines in the diagram

ν the number of loops; define

$$\rho = 2\nu - 1/2(\mathrm{L}+1)$$

$$\Delta_{\mathrm{L}} = \quad \text{Cayley / Gram}$$

the leading behavior of the diagram is given by

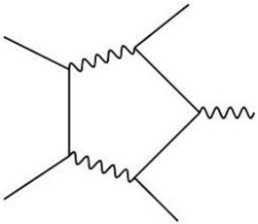
(N.B. Gram $\neq 0$)

$$\begin{aligned} \Delta_{\mathrm{L}}^{\rho} &\quad \text{for} \quad \rho < 0\,, \\ \Delta_{\mathrm{L}}^{k+1/2} &\quad \text{for} \quad \rho = k + \frac{1}{2}\,, \\ \Delta_{\mathrm{L}}^{k} \ln \Delta_{\mathrm{L}} &\quad \text{for} \quad \rho = k\,, \quad k \in \mathbb{Z}^{*}\,. \end{aligned}$$

Therefore for $\mathrm{L} = 2(2\nu + n) - 1$ and $n \in \mathbb{Z}^{+}$ the AT is a pole of order n for the amplitude, e.g. a simple pole for the one-loop pentagon, for two-loop diagrams with 9 propagators etc. In all other cases it is a branch point.

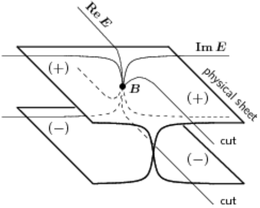
For $n \geq 6$ all the singularities of the one-loop n-point functions coincide with the singularities of the reduced, down to and including the pentagon, diagrams obtained from the main diagram (vanishing of Gram determinants if any of its principal minors vanish).

$\mathscr{G}$

$s \rightarrow$  $\equiv \quad \frac{1}{s - s_{\mathrm{AT}}}$

$\det \mathrm{C}$(ayley) is a quartic polynomial in s and we look for complex solutions in the variable s to the equation $\det \mathrm{C} = 0$. Since the AT for a pentagon is a pole this is exactly the situation where the AT could be misinterpreted as the peak due to an unstable particle.

Deciding whether it is a resonance would require establishing that the "pole" is on the second (unphysical) sheet.

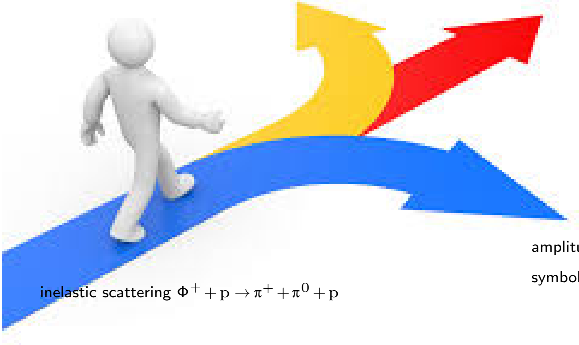


Projective geometry

Hodge structures

gravity duals

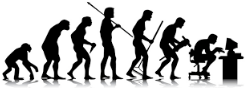
planar $N = 4$ super-Yang-Mills



amplituhedron

symbol entries

inelastic scattering $\Phi^{+} + \mathrm{p} \to \pi^{+} + \pi^{0} + \mathrm{p}$



PROLEGOMENA *Impact on physical observables*;

It should be emphasised that the presence of ATs depends on the structure of denominators of specific Feynman diagrams [3] but their numerical impact on the full amplitude also depends on numerators.

In this regard, we are assuming that the singularity spectrum of the S***-matrix is confined to the union of the singularity spectra of the Feynman integrals.***

and we proceed to construct the singularity spectra of the Feynman integrals Therefore, the scattering amplitude appears as the sum of infinitely many diagrams of increasing complexity and each diagram in principle can be completely investigated.

In principle there is no reason for the S-matrix to be the sum of the diagrams but ***we work under the assumption that the diagrams represent the local behavior of the amplitude*** and that the whole picture can be recovered by gluing together all these local behaviors.

[3] Because the vertices are point interactions, singularities in any local QFT are generated only by propagators.


Kershaw theorem
Coleman-Norton theorem
Cayley equations

Kershaw

The singular part of a scattering amplitude around its leading Landau singularity may be written as an algebraic product of the scattering amplitudes for each vertex of the corresponding Landau graph times a certain explicitly determined singularity factor which depends only on the type of singularity and on the masses and spins of the internal particles.

Furthermore, for a given set of external momenta which lie on the given physical Landau singularity there exists only one unique set of values for the internal momenta which satisfy the Landau equations.

It is worth noting that the consequences of the theorem have been reinterpreted by various authors in terms of multiple cuts on Feynman diagrams

Kershaw theorem $\Longleftarrow$ only one restriction need be observed: at least one internal line must have nonzero mass.

We are only interested in a  massive world

Technically speaking: further require that the given Landau singularity point does not also lie on the Landau curve of another graph, of which the given graph is a contraction.

Coleman and Norton

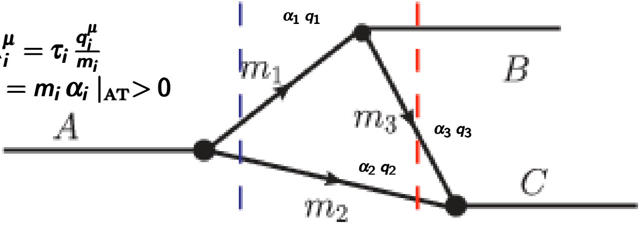
A Feynman amplitude has singularities on the physical boundary if and only if the relevant Feynman diagram can be interpreted as a picture of an energy- and momentum-conserving process occurring in space-time, with all internal particles real, on the mass shell, and moving forward in time.

As a by-product, the Feynman parameter associated with an internal line is identified (within a proportionality factor) with the time the particle exists between collisions, divided by its mass.

$$\Delta_i^\mu = \alpha_i \, q_i^\mu \, |_{\mathrm{AT}}$$

$$\Delta_i^\mu = \tau_i \, \frac{q_i^\mu}{m_i}$$

$$\tau_i = m_i \, \alpha_i \, |_{\mathrm{AT}} > 0$$

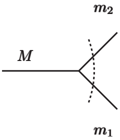


AT on first sheet $\alpha_i \in [0, 1]$

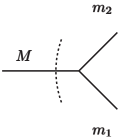
$\alpha_1 = 1 - x_1 \, , \, \alpha_2 = x_1 - x_2 \, , \, \alpha_3 = x_2$

$\sum_i \alpha_i = 1$

the graph can be read as a picture of classical particles moving in spacetime, with all momenta real and on mass shell, and all particles moving forward in time.

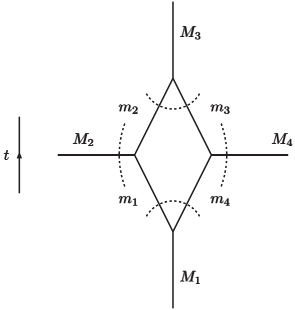


$M^2 > (m_1 + m_2)^2$

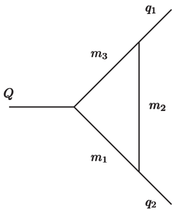


$M^2 < (m_1 - m_2)^2$

typical space-time diagram



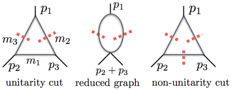
Triangle details: consider a triangle where the three external lines are off-shell (e.g. $\mathrm{H}^* \to \mathrm{W}^*\mathrm{W}^*$).



$$Q^2 = -s$$
$$q_i^2 = -M_i^2$$
$$Q = q_1 + q_2$$

3 and higher point functions can be cut im more that two pieces; putting all propagators on-shell corresponds to $\alpha_i \neq 0$ at the level of the Landau equations,

i.e. ATs go beyond the concept of unitarity cuts.



- We must have $s \geq (\mathrm{M}_1 + \mathrm{M}_2)^2$

From the Kershaw theorem we see that the ***physical-region Landau curve*** has six branches in the Q, q_1, q_2 space, i.e.

$$s > (m_1 + m_3)^2 \,, \quad \mathrm{M}_2^2 > (m_1 + m_2)^2 \,, \quad \mathrm{M}_1^2 < (m_2 - m_3)^2 \,,$$

with $Q^0 > 0$ and $q_2^0 > 0$.

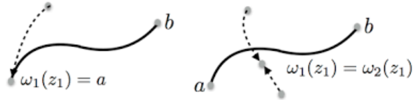
The other branches are obtained by cyclic permutations and by the overall reflection of the external momenta.

Example

an off-shell H with momentum Q going to off-shell Ws; internal lines are t, b quarks, i.e.

$$m_1 = m_3 = m_\mathrm{t} \,, \qquad m_2 = m_\mathrm{b} \,.$$

Furthermore, $\mathrm{M}_2 > m_\mathrm{t} + m_\mathrm{b}$ and $\mathrm{M}_1 < m_\mathrm{t} - m_\mathrm{b}$.



Pinchworld

homology

monodromy

$\mathscr{G}$

Consider a scalar, one-loop, N-point functions in d dimensions ($\mathrm{d} = 4 - \varepsilon$): external momenta will be labelled as $p_1, \dots p_{\mathrm{N}}$ and let us consider $\mathscr{P}_{\mathrm{N}}$ the set of the non-cyclic permutation of $(1, \dots \mathrm{N})$ with the first entry fixed. Vectors k_i are introduced according to the following convention:

$\mathrm{N} = 3$ there are two elements, i.e. $(1, 2, 3)$ and $(1, 3, 2)$. We define

$$(1,2,3) \quad \rightarrow \qquad k_1 = p_1,\, k_2 = p_2,$$
$$(1,3,2) \quad \rightarrow \qquad k_1 = p_1,\, k_2 = p_3.$$

$\mathrm{N} = 4$ There are three elements and we define

$$(1,2,3,4) \quad \rightarrow \qquad k_1 = p_1,\, k_2 = p_2,\, k_3 = p_3,$$
$$(1,2,4,3) \quad \rightarrow \qquad k_1 = p_1,\, k_2 = p_2,\, k_3 = p_4,$$
$$(1,3,2,4) \quad \rightarrow \qquad k_1 = p_1,\, k_2 = p_3,\, k_3 = p_2.,$$

$\mathrm{N} = 5$ There are twelve elements, etc.

Define a scalar integral as

$$S_{d;N}(w) = \frac{\mu^{\varepsilon}}{i\,\pi^2}\int d^{d}q\,\frac{1}{\prod_{i=0,N-1}[i]}, \qquad [i] = (q+k_0+\cdots+k_i)^2+m_i^2,$$

where $k_0 = 0$ and where w is an element of $\mathscr{P}_N$. Furthermore μ is the 't Hooft scale and $\varepsilon = d-4$. In parametric space we obtain

$$S_{dN}(w) = \left(\frac{\mu^2}{\pi}\right)^{2-d/2}\Gamma(N-\frac{d}{2})\,[N]_d(w),$$

where from the triangle to the hexagon we will use the following notations:

$$[3] \equiv C, \ldots [7] \equiv G.$$

We have

$$[\mathrm{N}]_{\mathrm{d}}(w) = \int dS_{\mathrm{N}-1}\, \mathrm{V}_{\mathrm{N}}^{\mathrm{d}/2-\mathrm{N}}(\mathrm{w}),$$

$$\mathrm{V}_{\mathrm{N}}(w) = x^t\, \mathrm{H}_{\mathrm{N}}(\mathrm{w})\, x + 2\, \mathrm{K}_{\mathrm{N}}^t(\mathrm{w})\, x + \mathrm{L}_{\mathrm{N}}(w), \quad X_{\mathrm{N}}(w) = -\mathrm{K}_{\mathrm{N}}^t(\mathrm{w})\, \mathrm{H}_{\mathrm{N}}^{-1}(\mathrm{w}),$$

$$\Delta_{\mathrm{N}}(w) = \mathrm{L}_{\mathrm{N}}(\mathrm{w}) - \mathrm{K}_{\mathrm{N}}^t(\mathrm{w})\, \mathrm{H}_{\mathrm{N}}^{-1}(\mathrm{w})\, \mathrm{K}_{\mathrm{N}}(\mathrm{w}),$$

where $\mathrm{H}_{ij} = -\, k_i \cdot k_j$; $\mathrm{G}_{\mathrm{N}} = \det \mathrm{H}_{\mathrm{N}}$ is the Gram determinant associated with the N-point function of argument $\mathrm{w} \in \mathscr{P}_{\mathrm{N}}$.

Furthermore, $\mathrm{K}_i = -1/2\,(m_i^2 - m_{i+1}^2 - k_i^2 - 2 \sum_{j=1,i-1} k_j \cdot k_i)$, $\mathrm{L} = m_1^2$ and

$$\int dS_{\mathrm{N}-1} = \int_0^1 dx_1 \int_0^{x_1} dx_2 \ldots \int_0^{x_{\mathrm{N}-2}} dx_{\mathrm{N}-1}\,.$$

N.B. X_{N} is the critical point of V_{N}, Δ_{N} is its critical value. A complex number is a critical value of a multivariate polynomial iff it is a root of the equation $\mathrm{discriminant\ of\ the\ polynomial} = 0$.

Let $\mathrm{M_N}$ be the $\mathrm{N} \times \mathrm{N}$ matrix

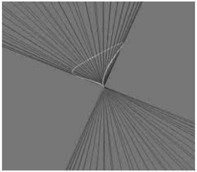
$$\mathrm{M_N} = \begin{pmatrix} \mathrm{H_N} & \mathrm{K_N} \\ \mathrm{K_N^{\mathit{t}}} & \mathrm{L_N} \end{pmatrix}$$

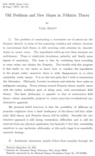
Then one can easily prove that

$$\Delta_\mathrm{N}(\mathrm{w}) = \frac{\mathrm{C_N(w)}}{\mathrm{G_N(w)}}\,, \qquad X^i_\mathrm{N} = \frac{\det \mathrm{M}_{(i,\mathrm{N})}}{\mathrm{G_N}}\,,$$

where $\mathrm{C_N} = \det \mathrm{M_N}$ is the so-called modified Cayley determinant of the diagram and we can write

$$\mathrm{V_N} \quad = \quad (x - X_\mathrm{N})^t\, \mathrm{H_N}\, (x - X_\mathrm{N}) + \Delta_\mathrm{N} = y^t\, \mathrm{M_N}\, y\,, \quad y^t = (x^t, 1)$$




Publ. RIMS, Kyoto Univ.
12 Suppl. (1977), 367-375.

Old Problems and New Hopes in S-Matrix Theory

by

Tullio REGGE*

1) The problem of constructing a convenient set of axioms for the S-matrix directly in terms of the scattering variables and without recourse to conventional field theory is still receiving wide attention by theoreticians in recent years. The ingredients which go into these attempts are well-known. There is relativistic invariance, unitarity, and finally some degree of analyticity. The hope is that by combining them according to some recipe one obtains the S-matrix. The trouble with this program is that really no one seems to know how to combine the ingredients in the proper order; moreover there is wide disagreement as to what analyticity really means. It is on this last point that I wish to concentrate the discussion. Obviously Lorentz invariance and unitarity have quite an ambiguous meaning. Every attempt toward S-matrix theory usually starts with the rather ambitious goal of doing away with conventional field theory. The basic philosophy is opposite to that of constructive field theory whose remarkable progress in recent years has overshadowed any alternative approach.

My personal belief however, is that the possibility of different approaches originates from a basic inadequacy of the theory and that ultimately field theory and S-matrix theory will be unified. Secondly the constructive approach is still facing tremendous difficulties and is still far removed from any physical application; but basically I feel that committing ourselves to any particular philosophy at this early stage is an essentially unsound strategy.

2) Analyticity statements usually follow from causality through the

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Standing on the shoulders of giants

Ferroglia et al.; Goria and P.

It is easily seen [4] that $\Delta_{\rm N} = 0$ induces a pinch on the integration contour at the point $x = X_{\rm N}$; therefore, if

$$\Delta_{\rm N} = 0, \qquad 0 < X_{\rm N,N-1} < \ldots < X_{\rm N,1} < 1,$$

we have the **leading singularity** (it represents a singular point of multiplicity two). Leading singularities of diagrams obtained by shrinking one (or more) line of the original diagram to a point give the sub-leading singularities.

[4] In general for a hypersurface $V(x_1 \ldots, x_n) = 0$ the singular points are those at which all the partial derivatives simultaneously vanish. The notion of singular points is a purely local property. The determination of the multiplicity of a singular point, is based on ascertaining which of the higher-order derivatives vanish at that point.

In the Cayley language the Landau equations for a general case can be written as follows. Consider the integral

$$\mathrm{I} = \int_{\mathrm{D}} \prod_{i=1}^{n} dx_i \, \mathrm{V}_{\mathrm{N},n}^{-\mu}(x_1 \ldots x_n \, ; w_1 \ldots w_k) \ ,$$

where

- V is a multivariate polynomial of degree N and an algebraic function of k-parameters $w_1 \ldots w_k$. Therefore $\mathrm{V} \in \mathbb{C}[x, w]$.
- D is the domain of integration.
- $\mathrm{V}_{\mathrm{N},n} = 0$ is the locus of the singularities of the integrand;

let $\mathscr{B}_j(x, w)$, $j = 1, \ldots, d$ be the set representing the boundary of D.

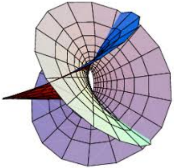
G

The necessary conditions for the leading singularities to occur when the hypercontour is pinched between the surfaces of singularity or meets a boundary variety are

- $\exists\ \alpha_i\,,\beta$, not all equal to zero and such that at the point $w_k = w_k^0$ and $x_k = x_k^0$ we have $\beta\,\mathrm{V}_{\mathrm{N},n} = 0$ and

$$\alpha_i\,\mathscr{B}_i\,(x_i\,,w_i) = 0, \qquad i = 1,\ldots,d\,,$$

$$\frac{\partial}{\partial x_i}\left[\sum_i \alpha_i\,\mathscr{B}_i\,(x\,,w) + \beta\,\mathrm{V}_{\mathrm{N},n}\,(x\,,w)\right] = 0, \qquad i = 1,\ldots,n\,.$$



Multi-loops

If D is a n-dimensional hypercube and

$$\mathrm{V}_{\mathrm{N},n} = \sum_{i=0}^{\mathrm{N}} \mathrm{V}_n^i \,, \qquad \mathrm{V}_n^i = \sum_{0 \le i_1 = \cdots i_n \le i} a^i_{i_1 \dots i_n} x_1^{i_1} \cdots x_n^{i_n} \,,$$

where the V_n^i are homogeneous polynomials and $\mathrm{V}_{\mathrm{N},n}$ is a generic polynomial in the ring of polynomials of degree N, it is convenient to determine the $(\mathrm{N}-1)^n$ n-tuples $X_1^i \dots X_n^i$ such that

$$\mathrm{V}_{\mathrm{N},n}\left(x_1 - X_1^i \dots x_n - X_n^i\right) = \Delta + \sum_{i=2}^{\mathrm{N}} \mathrm{V}_n^i \left(x_1 - X_1^i \dots x_n - X_n^i\right), \qquad i = 1 \dots (\mathrm{N}-1)^n \,,$$

so that the solutions of $\Delta(w_1 \dots w_k) = 0$, are the potential (leading) pinch singularities if $X_j^i \in \mathbb{R},\ 0 < X_j^i < 1\ \forall j$. For $\mathrm{V}_n^2 = \dots = \mathrm{V}_n^k = 0$ the singular point will have multiplicity $k+1$.

$\mathscr{G}$

One-loop summary: any one-loop diagram is specified by

① A multivariate polynomial $\mathrm{V}(x_1, \ldots, x_{\mathrm{N}-1})$

② $\mathrm{H_N}$, a $(\mathrm{N}-1) \times (\mathrm{N}-1)$ matrix whose determinant is the Gram determinant.

③ The critical point $\{X_1, \ldots, X_{\mathrm{N}-1}\} = \mathrm{X_N}$; $\mathrm{X_N^{ord}}$ is the set $\{(X_1, \ldots, X_{\mathrm{N}-1}) \in \mathbb{R}^{\mathrm{N}-1} \mid 0 < X_{\mathrm{N}-1} < \ldots X_1 < 1\}$.

④ The Bernstein-Sato-Tkachov factor Δ_N. The Bernstein theorem states that for any polynomial in $\mathbb{C}[x]$ there exists a non-zero polynomial $b(s) \in \mathbb{C}[s]$ (Berstein-Sato polynomial) and a differential operator $\mathrm{P}(s) \in D_n(s)$ such that $\mathrm{P}(s) \cdot f^{s+1} = b(s)\, f^s$. For one-loop diagrams Δ_N is the explicit form of b (Tkachov). Δ_N is the critical value of V.

⑤ The set of generalized Mandelstam invariants, I; $\mathrm{I_{phys}}$ is the set of invariants internal to the physical region [5].

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[5] "physical region" is identified with the phase space for the corresponding process, i.e. the physical region of a given process is the set of all real initial and final energy-momenta variables subject to the mass-shell conditions and to energy-momentum conservation. Solutions that correspond to points outside the physical region are on the wrong sheet.

Process $Q \to \sum_{i=1}^{4} q_i$ with $Q^2 = -s$, $q_i^2 = -\mathrm{M}_i^2$.

- set of invariant:

$$s_1 = -(Q-q_1)^2\,, \quad s_2 = -(Q-q_1-q_2)^2\,, \quad u_1 = -(Q-q_2)^2\,, \quad u_2 = -(Q-q_3)^2\,, \quad t_2 = -(Q-q_2-q_3)^2\,.$$

- Their limits, the physical region, are:

$$(\mathrm{M}_2+\mathrm{M}_3+\mathrm{M}_4)^2 \le s_1 \le (\sqrt{s}-\mathrm{M}_1)^2\,, \qquad (\mathrm{M}_3+\mathrm{M}_4)^2 \le s_2 \le (\sqrt{s_1}-\mathrm{M}_2)^2\,,$$

and $u_{1-} \le u_1 \le u_{1+}$ etc. where the limits can be written as $u_{1\pm} = u_{10} \pm du_1$ etc.

$$u_{10} = s + \mathrm{M}_2^2 - \frac{1}{2\,s_1}\,(s_1 - s_2 + \mathrm{M}_2^2)\,(s + s_1 - \mathrm{M}_1^2)\,, \quad \Delta u_1 = \frac{1}{2\,s_1}\,\lambda_1\,\lambda_2\,,$$

$$u_{20} = s + \mathrm{M}_3^2 - \frac{1}{2\,s_2}\,(s_2 + \mathrm{M}_3^2 - \mathrm{M}_4^2)\,(s + s_2 - s_2')\,, \quad \Delta u_2 = \frac{1}{2\,s_2}\,\lambda_3\,\lambda_4\,,$$

$$t_{20} = u_1 + \mathrm{M}_3^2 - \frac{1}{2\,s}\,(s - u_2 + \mathrm{M}_3^2)\,(s + u_1 - \mathrm{M}_2^2) - \frac{1}{2}\,\frac{\xi_2\eta_2}{s}\,\lambda_5\,\lambda_6\,, \quad dt_2 = \frac{1}{2\,s}\,(1-\xi_2^2)^{1/2}\,(1-\eta_2^2)^{1/2}\,\lambda_5\,\lambda_6\,.$$

$$\begin{aligned}
\lambda_1 &= \lambda^{1/2}(s\,,s_1\,,\mathrm{M}_1^2)\,, \quad \lambda_2 = \lambda^{1/2}(s_1\,,s_2\,,\mathrm{M}_2^2)\,, \quad \lambda_3 = \lambda^{1/2}(s\,,s_2\,,s_2')\,,\\
\lambda_4 &= \lambda^{1}2(s_2\,,\mathrm{M}_3^2\,,\mathrm{M}_4^2)\,, \quad \lambda_5 = \lambda^{1/2}(s\,,u_1\,,\mathrm{M}_2^2)\,, \quad \lambda_6 = \lambda^{1/2}(s\,,u_2\,,\mathrm{M}_3^2)\,.
\end{aligned}$$

$$s_2' = s - s_1 + s_2 - u_1 + \mathrm{M}_1^2 + \mathrm{M}_2^2\,, \qquad t_1' = \mathrm{M}_2^2\,.$$

The variables ξ_2 and η_2 can be found in Kumar:1970cr with full details on the calculation of the phase space.

$\mathbb{R}$eal (internal) masses:

Assume that all masses are real ($\{\Gamma\} = 0$), the physical-region LLS (or physical-region anomalous threshold) is given by

$$\Delta_{\mathrm{N}}\Big|_{\{\Gamma\}=0} = 0\,, \quad \mathrm{X_N} = \mathrm{X_N^{ord}}\,, \quad \{\mathrm{I}\} = \{\mathrm{I}\}_{\mathrm{phys}}\,.$$

$\{X_i\}$ gives the critical point of the diagram and $\{X_i - X_{i+1}\}$ gives the ratio proper time / mass of the internal particles.

There are cases where the first two conditions are satisfied but Mandelstam invariants are moved to their complex plane. Nevertheless, their real part can be inside the physical region with a tiny imaginary part; therefore they can be very close to the boundary.

C omplex (internal) masses:

When internal masses are made complex, i.e. $m_i^2 \to m_i^2 - i\,\Gamma_i\, m_i$, singularities move into the complex x-space. We are nevertheless interested in the following configurations:

$$\mathrm{Re}\,\Delta_{\mathrm{N}} \approx \Delta_{\mathrm{N}}\Big|_{\{\Gamma\}=0} \approx 0\,, \qquad \mathrm{Im}\,\Delta_{\mathrm{N}} \ll 1\,,$$

$$\{\mathrm{Re}\,X_i\} = \{\mathrm{Re}\,X_i\}^{\mathrm{ord}}\,, \qquad \mathrm{Im}\,X_i \ll 1\,,$$

with $\{\mathrm{I}\} = \{\mathrm{I}\}_{\mathrm{phys}}$. The introduction of complex masses regularizes the singularity since, in general, $\mathrm{Im}\,\Delta_{\mathrm{N}} \neq 0$;

Peierls **0** s

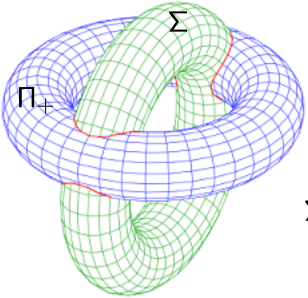
They are defined by

$$\mathrm{Re}\,\Delta_{\mathrm{N}} \quad = \quad \mathrm{Im}\,\Delta_{\mathrm{N}} = 0$$

and we look for a set of "real" invariants that satisfy this equation, possibly within the physical region and with $\{\mathrm{Re}\,X_i\} = \{\mathrm{Re}\,X_i\}^{\mathrm{ord}}$.

can be seen by considering a simple example:

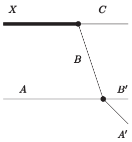
$$F(z_1, z_2) = \int_{-1}^{+1} dx \left[(x - z_1)^2 - z_2^2\right]^{-1}$$



$$\Sigma = \{z_2 = 0\}$$

$$\Pi_+ = \{z_1 \,|\, \mathrm{Im}\, z_1 = 0, |\,\mathrm{Re}\, z_1\,| < 1\}$$

If we follow a path on Π_+ and approach Σ a pinch will appear; starting with a point in Σ and not in Π_+ and following any path on Σ will not give a pinch singularity.



From Coleman-Norton and Kershaw theorems we immediately realize that:

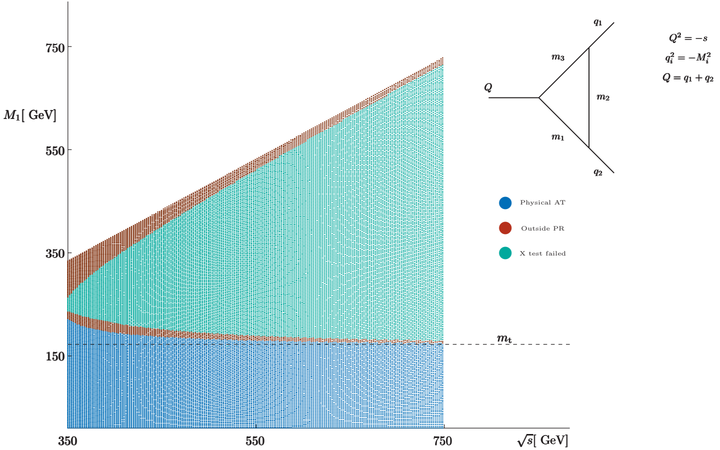
① a physical-region singularity requires a theory with a hierarchy of heavy masses. Identical masses in a vertex must be avoided.

② If we stay within the SM this means that we are limited to consider only two heavy particles, the t-quark and the W-boson.

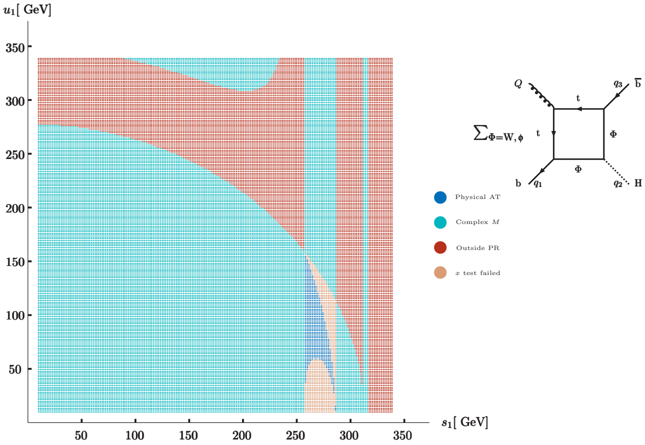
③ Anomalous thresholds in the SM prefer the so-called "off-shell" region, e.g. gg producing an off-shell Higgs (with a virtuality greater that $2\,m_{\mathrm{t}}$) subsequently "decaying" into four fermions. The situation changes when we consider BSM models.

The exceptional case with a vanishing modified Cayley determinant hardly appears in applications

that's not really true: you must unlearn what you have learned. You will find only what you bring in.



$m_1 = m_3 = m_\mathrm{t}$, $m_2 = m_\mathrm{b}$. For given values of $\sqrt{s}$ and M_1 the value of M_2 corresponding to the AT is computed.



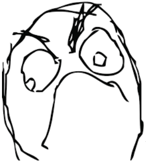
A scan in the $s_1 - u_1$ plane searching for a physical-region AT in $\mathrm{gg} \to \overline{\mathrm{b}}\,\mathrm{b}\mathrm{H}(\mathrm{HH})$ for $\sqrt{s} = 350$ *GeV*.

$Q = q_1 + q_2 + q_3$, with $Q^2 = -s$ and $q_i^2 = -\mathrm{M}_i^2$

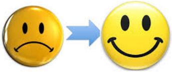
$s_1 = -(Q - q_1)^2\,,\; u_1 = -(Q - q_2)^2$

$(\mathrm{M}_2 + \mathrm{M}_3)^2 \le s_1 \le (\sqrt{s} - \mathrm{M}_1)^2\,,\; u_{1-} \le u_1 \le u_{1+}$

$u_{1\pm} = s + \mathrm{M}_2^2 - \frac{1}{2\,s_1}\left[(s_1 + \mathrm{M}_2^2 - \mathrm{M}_3^2)(s + s_1 - \mathrm{M}_1^2) \mp \lambda^{1/2}(s_1\,, \mathrm{M}_2^2\,, \mathrm{M}_3^2)\,\lambda^{1/2}(s\,, s_1\,, \mathrm{M}_1^2)\right]$

 ? ***too much of offshellness?***

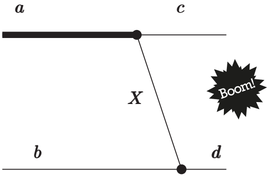
Yes, physical-region ATs are exceptional for SM, on-shell, LHC physics but more frequent in the so-called "off-shell" LHC physics; the reason for that is immediately seen in the context of the Coleman-Norton and Kershaw theorems, i.e. ***not enough heavy masses in the SM to be in one of the*** $6(14,\dots)$ ***branches of the physical-region Landau curve for a triangle (box,…) diagram.***



More SM and BSM to come

Benchmark a

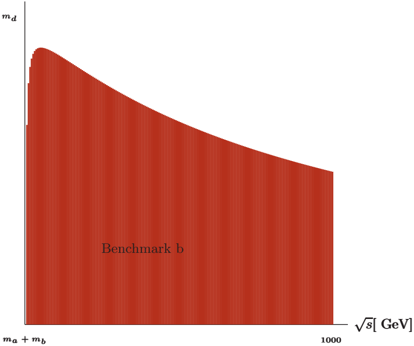
$m_a = 500\ GeV$, $m_b = 1\ GeV$, $m_c = 2\ GeV$, $m_d = 100\ GeV$

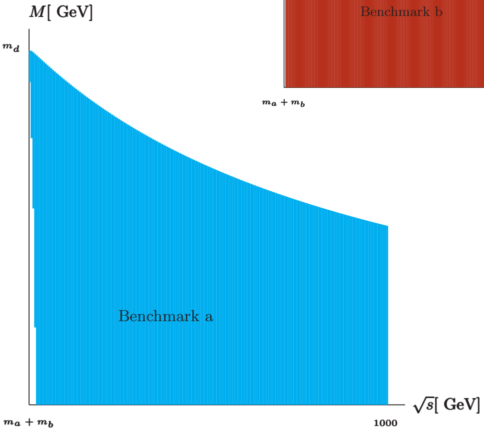


Benchmark b

$m_a = 500\ GeV$, $m_b = 1\ GeV$, $m_c = 2\ GeV$, $m_d = 10\ GeV$

$t = M_{\mathrm{X}}^2$ inside phys. reg. ($\notin$ SM)

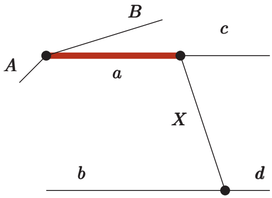

m_d
Benchmark b
$m_a + m_b$
1000
$\sqrt{s}$[GeV]


M[GeV]
m_d
Benchmark a
$m_a + m_b$
1000
$\sqrt{s}$[GeV]

No need to worry, need to remember

a is not an asymptotic state

This is the full process $\Longrightarrow$



Let $\boldsymbol{t} = -(\boldsymbol{p_b} - \boldsymbol{p_d})^2$, for stable A, $B \Rightarrow$ $\boldsymbol{t} < \boldsymbol{0}$

The fuse has gone out

AT induced by radiation

- No AT for $\mathrm{X} \to \bar{\mathrm{f}}\mathrm{f}$, where X can be an on-shell BSM Higgs boson, the off-shell SM Higgs boson or an off-shell Z boson. However, considering the processes

$$\mathrm{X} \to \overline{\mathrm{b}}\mathrm{b}\mathrm{g}\,, \qquad\qquad \mathrm{X} \to \bar{\mathrm{f}}\mathrm{f}\gamma\,,$$

- there is a class of one-loop diagrams which admits a physical-region AT.

$\overline{\mathrm{b}}\mathrm{bg}(\mathrm{g})$. If s denotes the X virtuality, there a small window between $\sqrt{s} = 345.5$ *GeV* where the AT corresponds to $\mathrm{M}(\mathrm{bg}) = 264.81$ *GeV* and $\sqrt{s} = 370$ *GeV* where the AT corresponds to $\mathrm{M}(\mathrm{bg}) = 252.90$ *GeV*.

AT induced by radiation

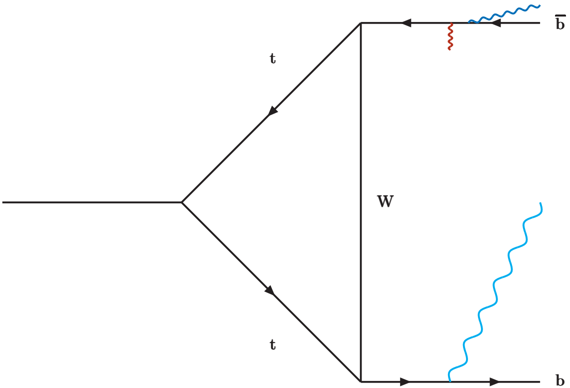
- $e^+e^-\gamma$: for this process we can have a $W-\nu-W$ triangle: in this case $\sqrt{s}$ must be above $2\,M_W$ for a physical-region AT. The physical-region AT starts at $\sqrt{s} = 161.05$ *GeV* with $M(e^+\gamma) = 110.64$ *GeV* and approaches $M(e^+\gamma) = M_W$ for very large values of $\sqrt{s}$.

- 

Due to the small value of m_e the inclusion of NNLO terms changes drastically the result, i.e. the NNLO contribution is six orders of magnitude larger than the LO term at $\sqrt{s} = 170$ *GeV*. In this case there is a peak at the AT.

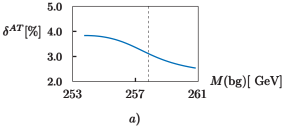
- We can also have a $Z-e-Z$ triangle: in this case $\sqrt{s}$ must be above $2\,M_Z$. The physical-region AT starts at $\sqrt{s} = 182.38$ *GeV* with $M(e^+\gamma) = 128.96$ *GeV* and approaches $M(e^+\gamma) = M_Z$ for very large values of $\sqrt{s}$.

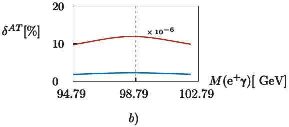
Therefore, for e^+e^-, $\overline{q}q$ and gg initial states there are amplitudes where the initial state couples to a neutral object which can couple to WW or $\bar{t}t$. In these cases we can have $\overline{b}bg$ and $\bar{f}f\gamma$ final states ***producing a singularity at large*** $f(\bar{f})-g(\gamma)$ ***invariant masses, on top of the more familiar infrared and collinear ones.***

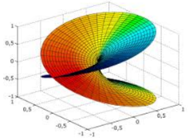


H from gg, Z from e^+e^-, $H(Z) \rightarrow \bar{t}t$ allowed $t \rightarrow Wb(\gamma)$ allowed, $Wt \rightarrow b\gamma$ allowed

Figure : AT induced radiative corrections for a) $\mathrm{H}^* \to \overline{\mathrm{b}}\mathrm{b}\mathrm{g}$ and b) $\mathrm{H}^* \to \mathrm{e}^+\mathrm{e}^-\gamma$. The H virtuality is a) $\sqrt{s} = 350$ *GeV* and b) $\sqrt{s} = 170$ *GeV*.







Fate of the AT: there is a general result, for processes requiring more and more invariants the fully inclusive observables become less and less sensitive to ATs.

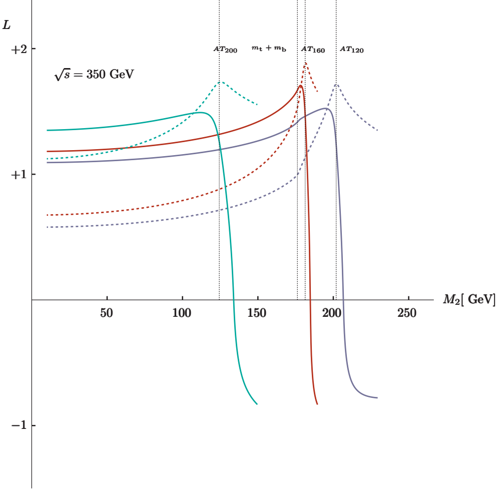
$$\mathrm{H}(Q) \to \mathrm{W}^+(q_1+q_2) + \mathrm{W}^-(q_3+q_4) \to \underbrace{\nu_\mu(q_1) + \mu^+(q_2)}_{\mathrm{M}_1} + \underbrace{\mathrm{e}^-(q_3) + \overline{\nu}_\mathrm{e}(q_4)}_{\mathrm{M}_2},$$

- with $Q^2 = -s$ and light fermion masses are neglected. The full process is given in terms of s and 5 additional Mandelstam invariants,

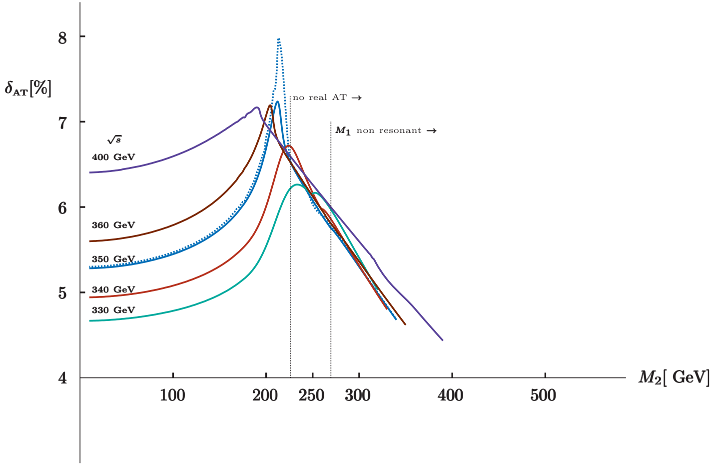
$$\begin{aligned} s_1 &= -(q_2+q_3+q_4)^2, \qquad s_2 = -(q_3+q_4)^2, \\ u_1 &= -(q_1+q_3+q_4)^2, \qquad u_2 = -(q_1+q_2+q_4)^2, \\ t_2 &= -(q_1+q_4)^2. \end{aligned}$$

- We study the $d\Gamma/d\mathrm{M}_2$ distribution,

$$\frac{d\Gamma}{d\mathrm{M}_2} = \frac{d\Gamma}{d\mathrm{M}_2}\Big|_{\mathrm{LO}} + \frac{d\Gamma}{d\mathrm{M}_2}\Big|_{\mathrm{NLO},\mathrm{AT}} + \frac{d\Gamma}{d\mathrm{M}_2}\Big|_{\mathrm{NLO},\mathrm{rest}}.$$



The log-modulus transformation of $s\,C_0$ as a function of M_2 for $\sqrt{s} = 350\ GeV$, and $\mathrm{M}_1 = 120\ GeV$ (blue), $\mathrm{M}_1 = 160\ GeV$ (red) and $\mathrm{M}_1 = 200\ GeV$ (emerald). Solid curves give the real part while dashed curves give the imaginary part. $\mathrm{L}(x) = \mathrm{sign}(x)\,\frac{\ln(1+|x|)}{\ln 10}$



M_1 is non resonant if $\mathrm{M}_2 > \sqrt{s} - M_{\mathrm{W}}$ and above a certain value for M_2 there is no AT corresponding to a real value of M_1. For instance, for $\sqrt{s} = 350$ *GeV* this happens above $\mathrm{M}_2 \approx 226$ *GeV*. The (blue) dashed line corresponds to $\sqrt{s} = 350$ *GeV* and $\Gamma_{\mathrm{t}} \to \Gamma_{\mathrm{t}}/100$, showing the effect of the AT.

Furthermore, folding:

$e^+e^- \to X$; "radiator approach". the hard scattering cross section is convoluted with initial state QED radiation,

$$\sigma(s) \;=\; \int_0^1 dz\, H(z,s)\, \widehat{\sigma}((1-z)\,s)\,,$$

where $s = 4\,E^2$, E being the beam energy. It will be enough to assume the so-called virtual-soft approximation where $H = H_0\,\beta\,z^{\beta-1}\,,\ 0 < \beta \ll 1$. EX:

$$F(s) \;=\; \int_0^1 dz\, H(z\,,\,s)\, f(s)\,,\ f(s) = \Big|\int_0^1 dx\, \chi^{-1}(x\,,\,s)\Big|^2\,,$$

where $\chi = s\,x^2 + (m_2^2 - m_1^2 - s)\,x + m_1^2$.

$$s_{\rm p} \;=\; (m_1 - m_2)^2 \,,\; s_{\rm t} = (m_1 + m_2)^2 \,,\; \lambda = (s - s_{\rm p})\,(s - s_{\rm t}) \,.$$

① Behavior of $f(s)$ around the normal threshold $s_{\rm t}$:

$$f(s) \;=\; 4\,\frac{\pi^2}{\lambda} + \text{ reg. terms} \qquad s \to s_{\rm t} \,,$$

The function has a simple pole at the normal threshold.

② In terms of the folding this means that $\mathrm{F}(s) \sim (s_{\rm t} - s)^{\beta - 1}$ which is integrable and can be used in the convolution with the beam energy spread.

Cayley vs. Gram
Landau vs. non-Landau

$$\mathrm{I} \;=\; \int_0^1 dx \int_0^x dy \left[x^2 - \lambda\, y^2 + 2\,(a\,x - \lambda\, b\,y) + \mathrm{L}\right]^{-1}.$$

- In this case we derive $X_1 = -a, X_2 = b$, $\boxed{\mathrm{G}} = \lambda$ and $\boxed{\mathrm{C}} = \mathrm{L} - a^2 - \lambda\, b^2$. We assume that $0 \le b \le -a \le 1$ and derive the usual result

$$\mathrm{I} \;\sim\; \ln \frac{\boxed{\mathrm{C}}}{\boxed{\mathrm{G}}}\,, \qquad \Delta = \frac{\boxed{\mathrm{C}}}{\boxed{\mathrm{G}}} \to 0\,.$$

- However, if we take the limit $\lambda \to 0$ first then $\boxed{\mathrm{G}} = 0$. In this case we obtain

$$\mathrm{I}\,|_{\lambda=0} \;\sim\; (\mathrm{L} - a^2)^{-1/2}\,, \qquad \mathrm{L} \to a^2\,,$$

  where $\mathrm{L} - a^2$ is the Cayley determinant evaluated at $\boxed{\mathrm{G}} = 0$.

- Therefore, the behavior is $\boxed{\mathrm{C}}^{-1/2}$ and not $\ln \Delta$.

Are complex masses enough?

- In the complex mass scheme we have $\Delta_{\mathrm{N}}, X_i \in \mathbb{C}$. With no loss of generality, we fix $\mathrm{N} = 4$ and consider a general process $Q \to q_1 + q_2 + q_3$ where $Q^2 = -s$ and $q_i^2 = -\mathrm{M}_i^2$. Let s_1 and u_1 be the two independent invariants.

- Consider the following system of equations:

$$\mathrm{Re}\,\Delta_4 = 0 \qquad \mathrm{Im}\,\Delta_4 = 0$$

$$\mathrm{Im}\,X_i = 0 \qquad i = 1, \dots, 3$$

- Therefore we have 5 equations in 6 unknowns, $s, s_1, u_1, \mathrm{M}^2_{1,2,3}$; a solution will give a surface parametrized by $s_1 = s_1(s), u_1 = u_1(s)$ etc.

- If the real part of the X_i, evaluated at the solution, is ordered then ***we may have a pinch singularity even in the complex mass scheme***. The square of the box will be non-integrable.

These conditions are very difficult to satisfy!

- ***The main motivation was to find observable effects, e.g.. peaks in distributions; indeed, resonances are normally observed as peaks in certain invariant mass distributions. However, is a peak necessarily due to the presence of a resonance? Are there peaks produced by kinematic singularities?***

- ***It is well known that scattering amplitudes also possess singularities corresponding to more complicated types of particle exchanges: these are indeed the Landau singularities.***

  A well-know example is that there has been interest for many years in trying to use the triangle singularity to explain certain enhancements in strongly interacting three-particle final states.

$\mathscr{G}$

- *(at LHC) SM peaks, when present, are marginal and radiative corrections induced by physical-region ATs are well under control once regularized with the complex mass scheme; nevertheless they should be taken into account in estimating the missing higher order uncertainty.*

  When assessing such results, it should be borne in mind that Yukawa suppressed LO processes can be heavily influenced by NLO corrections, i.e. NLO is the first relevant term.

BSM, another story

- Any BSM theory with an heavy neutral Higgs boson (H) and a charged one $(\mathrm{H}^{\pm})$ satisfying $M_{\mathrm{H}} > 2\,M_{\mathrm{H}^{\pm}}$ and $M_{\mathrm{H}^{\pm}} > M_{\mathrm{W}} + M_{\mathrm{Z}}$ will have an AT in the decay $\mathrm{H} \to \mathrm{ll}\nu\nu$.
- One could even imagine a situation with a light Higgs boson h and 3 heavy Higgs bosons, $\mathrm{H}_{1,2,3}$, where $\mathrm{M}_{\mathrm{H}_1} > 2\,\mathrm{M}_{\mathrm{H}_2}$, $\mathrm{M}_{\mathrm{H}_2} > \mathrm{M}_{\mathrm{H}_3} + \mathrm{M}_{\mathrm{h}}$ and $\mathrm{M}_{\mathrm{H}_3} > 3\,\mathrm{M}_{\mathrm{h}}$ giving an AT in the pentagon corresponding to $\mathrm{H}_1 \to 6\,\mathrm{h}$.

Thank you for your attention


Backup Slides

of leading behavior

- To extract the leading behavior of a triangle around its leading Landau singularity we introduce

$$C_{\square}\,(p_1, p_2\,;\, m_1, m_2, m_3) \quad = \quad C_0\,(p_1, p_2\,;\, m_1, m_2, m_3) + C_0\,(p_2, p_1\,;\, m_3, m_2, m_1)\,.$$

- If the first C_0 in the r.h.s. is singular then the second is regular and

$$C_0\,(p_1, p_2\,;\, m_1, m_2, m_3) \quad \sim \quad C_{\square}\,(p_1, p_2\,;\, m_1, m_2, m_3)\,.$$

- Given the quadratic forms

$$V_3\,(x_1, x_2) \quad = \quad x^t\,H\,x + 2\,K^t\,x + L,$$

$$\begin{aligned} V^{(1)}(x) &= V_3(0,x), \quad V^{(2)}(x) = V_3(x,0), \\ \overline{V}^{(1)}(x) &= V_3(1,x), \quad \overline{V}^{(2)}(x) = V_3(x,1), \end{aligned}$$

- consider two-dimensional bubbles

$$B_2^{(i)} \quad = \quad \int_0^1 dx\,\frac{1}{V^{(i)}(x)}, \qquad \overline{B}_2^{(i)} = \int_0^1 dx\,\frac{1}{\overline{V}^{(i)}(x)}.$$

- We find

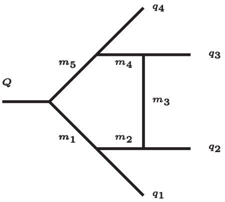
$$C_\Box \quad \sim \quad -\frac{1}{2}\ln B_3 \sum_{i=1,2}\Big[X_i\, B_2^{(i)} + (1-X_i)\,\overline{B}_2^{(i)}\Big],$$

for $B_3 \to 0$.

- The explicit result is

$$\begin{aligned} C_\Box \quad &\sim \quad -\frac{1}{4}\,(-G_3)^{-1/2}\ln B_3 \\ &\times \quad \Big[\ln\frac{H_{11}+H_{12}+L+2\,K_1+K_2+(1-X_1)\,(-G_3)^{1/2}}{H_{11}+H_{12}+L+2\,K_1+K_2-(1-X_1)\,(-G_3)^{1/2}} \\ &+ \quad \ln\frac{H_{22}+H_{12}+L+2\,K_2+K_1+(1-X_2)\,(-G_3)^{1/2}}{H_{22}+H_{12}+L+2\,K_2+K_1-(1-X_2)\,(-G_3)^{1/2}} \\ &+ \quad \ln\frac{L+K_2+X_1\,(-G_3)^{1/2}}{L+K_2-X_1\,(-G_3)^{1/2}} + \ln\frac{L+K_1+X_2\,(-G_3)^{1/2}}{L+K_1-X_2\,(-G_3)^{1/2}}\Big]. \end{aligned}$$

- For generalized triangles we obtain $C_\Box(i,j) \sim X_1^i\, X_2^j\, C_\Box$, as expected by the fact that the AT is a pinch singularity.



- Invariants: $Q^2 = -s$, $q_i^2 = -\mathrm{M}_i^2$

$$s_1 = -(Q-q_1)^2\,,\ s_2 = -(Q-q_1-q_2)^2\,,\ u_1 = -(Q-q_2)^2\,,$$
$$u_2 = -(Q-q_3)^2\,,\ t_2 = -(Q-q_2-q_3)^2\,.$$

- Process: $\mathrm{gg} \to \mathrm{g} \to \overline{\mathrm{b}}\mathrm{b}\mathrm{e}^-\mathrm{e}^+\gamma$. q_3 is a neutrino while q_2 is the momentum of a pair, electron-photon. In the limit of zero widths the equation $\Delta_5 = 0$ is quadratic in t_2, the $\overline{\mathrm{b}}\mathrm{b}$ (squared) invariant mass.

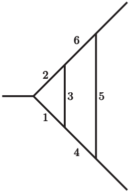
𝒢

fixed					solution
M_2[*GeV*]	$\sqrt{s_1}$[*GeV*]	$\sqrt{s_2}$[*GeV*]	$\sqrt{u_1}$[*GeV*]	$\sqrt{u_2}$[*GeV*]	$\sqrt{t_2}$[*GeV*]
97.40	260.47	139.17	201.03	275.84	122.53
"	"	"	203.55	260.91	89.06
"	"	"	"	265.74	103.56
"	"	"	"	270.56	117.37
"	"	"	"	275.39	130.91
"	"	147.14	213.31	266.39	121.68
"	"	"	"	271.07	138.30
"	"	"	215.30	255.98	98.96
"	"	"	"	260.89	113.19

$\mathscr{G}$

$$\alpha_1\,(q_1^2+m_1^2)=0\,,\qquad \alpha_2\,((q_1+P)^2+m_2^2)=0\,,\qquad \alpha_3\,((q_1-q_2)^2+m_3^2)=0\,,$$
$$\alpha_4\,(q_2^2+m_4^2)=0\,,\qquad \alpha_5\,((q_2+p_1)^2+m_5^2)=0\,,\qquad \alpha_6\,((q_2+P)^2+m_6^2)=0\,,$$

$$\begin{aligned} \alpha_1 q_{1\mu}+\alpha_2(q_1+P)_\mu+\alpha_3(q_1-q_2)_\mu &= 0\,,\\ -\alpha_3(q_1-q_2)_\mu+\alpha_4\,q_{2\mu}+\alpha_5\,(q_2+p_1)_\mu+\alpha_6\,(q_2+P)_\mu &= 0\,, \end{aligned}$$



$[1]=q_1^2+m_1^2$
$[2]=(q_1+P)^2+m_2^2$
$[3]=(q_1-q_2)^2+m_3^2$
$[4]=q_2^2+m_4^2$
$[5]=(q_2+p_1)^2+m_5^2$
$[6]=(q_2+P)^2+m_6^2$

- LLS occurs for $\alpha_i \neq 0, \forall i$. Multiply the two equations by $q_{1\mu}$, $q_{2\mu}$, $p_{1\mu}$ and P_μ, respectively. This gives an homogeneous system of eight equations. If all α_i are different from zero we may use

$$q_1^2 = -m_1^2 \qquad q_2^2 = -m_4^2\,, \qquad q_1 \cdot q_2 = \frac{1}{2}\,(m_3^2 - m_1^2 - m_4^2)\,,$$
$$q_1 \cdot P = -\frac{1}{2}\,(P^2 - m_1^2 + m_2^2)\,, \qquad q_2 \cdot p_1 = -\frac{1}{2}\,(p_1^2 - m_4^2 + m_5^2)\,, \qquad q_2 \cdot P = -\frac{1}{2}\,(P^2 - m_4^2 + m_6^2)\,.$$

- Compatibility requires a set of relations among $P^2, p_{1,2}^2$ and internal masses.

- Select $m_3 = 0$ (gluon, photon), $m_5 = m_{\rm b}$ and the remaining masses equal to $m_{\rm t}$ then a non trivial solution ($\alpha_i \neq 0, \forall i$) occurs iff

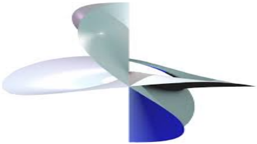
$$s = -P^2 = 4\,m_{\rm t}^2\,,\ \mathrm{M}_1^2 = -p_1^2 = (m_{\rm t} \pm m_{\rm b})^2\,,\ \mathrm{M}_2^2 = -p_2^2 = (m_{\rm t} \mp m_{\rm b})^2\,,$$

i.e. exactly at the boundary of phase space.

- The solution is $\alpha_{1,3,4,5}$ arbitrary and $\alpha_2 = \alpha_1$, $m_{\rm t}\,\alpha_6 = m_{\rm t}\,\alpha_4 - m_{\rm b}\,\alpha_5$. This solution includes the case $\alpha_3 = 0$.
- If $m_3 = M_{\rm Z}$ we obtain $s = 4\,m_{\rm t}^2 - M_{\rm Z}^2$ and

$$
\begin{aligned}
& 2\,m_{\rm t}^2\,({\rm M}_1^2 + {\rm M}_2^2)^2 - \Big[8\,(m_{\rm t}^2 + m_{\rm b}^2)\,m_{\rm t}^2 - (3\,m_{\rm t}^2 + m_{\rm b}^2)\Big]\,({\rm M}_1^2 + {\rm M}_2^2) - M_{\rm Z}^2\,{\rm M}_1^2\,{\rm M}_2^2 \\
- \quad & \Big[(m_{\rm t}^2 + m_{\rm b}^2)^2 + 4\,m_{\rm t}^2\,(m_{\rm t}^2 + 2\,m_{\rm b}^2)\,M_{\rm Z}^2 + (m_{\rm t}^2 + m_{\rm b}^2)\,M_{\rm Z}^4 + 8\,(m_{\rm t}^2 + m_{\rm b}^2)^2\,m_{\rm t}^2 = 0\,.
\end{aligned}
$$

- It is easily seen that the last equation does not have real solutions for ${\rm M}_{1,2}$ for physical values of $M_{\rm Z}, m_{\rm t}$ and $m_{\rm b}$.



More on n-tple points of algebraic surfaces

- Given

$$\mathrm{P}_{3,1}(x) \quad = \quad x^3 + a_2\, x^2 + a_1\, x + a_0$$

one introduces

$$q = \frac{1}{3}\, a_1 - \frac{1}{9}\, a_2^2, \;\; r = \frac{1}{6}\, (a_1\, a_2 - 3\, a_0) - \frac{1}{27}\, a_2^3,$$

- The following identities hold

$$\begin{aligned}
\mathrm{P}_{3,1} &= (y + 2\, q^{1/2})(y - q^{1/2})^2 - 2(r + q^{3/2}), \text{ for } q \geq 0, r < 0,\\
\mathrm{P}_{3,1} &= (y - 2\, q^{1/2})(y + q^{1/2})^2 - 2(r - q^{3/2}), \text{ for } q \geq 0, r \geq 0,\\
\mathrm{P}_{3,1} &= (y + 2 \mid q \mid^{1/2})(y - \mid q \mid^{1/2})^2 - 2(r + \mid q \mid^{3/2}), \text{ for } q < 0, r < 0,\\
\mathrm{P}_{3,1} &= (y - 2 \mid q \mid^{1/2})(y + \mid q \mid^{1/2})^2 - 2(r - \mid q \mid^{3/2}), \text{ for } q < 0, r \geq 0,
\end{aligned}$$

where $y = x + \frac{1}{3}\, a_2$.

𝒢

- Generalization to an arbitrary bivariate cubic form

$$\mathrm{P}_{3,2}(x_1, x_2) \quad = \quad \sum_{i=0}^{3} \sum_{j=0}^{3-i} c_{ij}\, x_1^i\, x_2^j,$$

- perform a linear transformation, $x_i' = x_i - X_i$, requiring that the linear terms vanishes, i.e. $c_{01} = c_{10} = 0$. In this case

$$\mathrm{P}_{3,2}(x_1, x_2\ ) \quad = \quad \mathrm{P}_2^3(x_1 - X_1\,, x_2 - X_2) + \mathrm{P}_2^2(x_1 - X_1\,, x_2 - X_2) + \Delta$$

- where the point $x_i = X_i$ corresponds to the vanishing of $\mathrm{P}_{3,2}$ and of its two first derivatives for $\Delta = 0$. Δ is therefore the natural extension of the Bernstein - Sato factor to cubic polynomials. However, opposite to the quadratic case, here there is no guarantee that the X_i are real. Note that Δ is an algebraic invariant of the singularity.