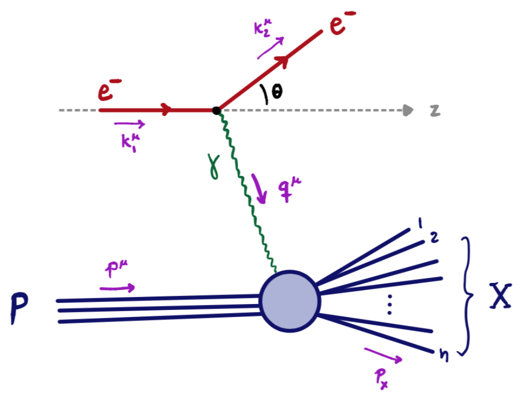
Elementary Particle Theory – PHY452

Fall Semester 2020

Exercise Sheet 8

Exercise 1: Electron-proton deep inelastic scattering

The Deep inelastic scattering (DIS) $e^- P \to e^- X$ is a QED process described by the following kinematic configuration



where X denotes any number of final state hadrons with total momentum $p_X = p + q$. The momentum transfer is defined as $Q^2 = -q^2$ and the momentum fraction of the proton is given by the Björken variable $x = Q^2/2p \cdot q$.

a) Derive an explicit parametrization for the massless momenta k_1 and k_2, assuming the proton at rest $p^\mu = (M, 0, 0, 0)$ and $\vec{k}_1$ chosen along the z axis. Denote the incoming and outgoing electron energies as k_1^0 and k_2^0, respectively, and the remaining angular degrees of freedom θ and ϕ.

b) Perform a change of coordinates $(k_2^0, \theta) \to (x, Q^2)$ keeping fixed k_1^0 and show that:

$$\mathrm{D}k_2 = \frac{y}{16\pi^2 x}\, \mathrm{d}x\, \mathrm{d}Q^2 \,. \tag{1}$$

where $y := (p \cdot q)/(p \cdot k_1)$ and $\mathrm{D}k := \frac{\mathrm{d}^3 k}{(2\pi)^3 2k^0}$.

c) Derive an inequality from momentum conservation to show that $x \in [0, 1]$.

d) Show that for massless electrons, the DIS cross-section in the lab frame can be brought to the form

$$\frac{\mathrm{d}\sigma}{\mathrm{d}x \mathrm{d}Q^2} = \frac{\alpha^2 y^2}{Q^6} [k_1^\mu k_2^\nu + k_2^\mu k_1^\nu - (k_1 \cdot k_2) g^{\mu\nu}] W_{\mu\nu} \,, \tag{2}$$

where $W^{\mu\nu}$ is a purely hadronic tensor and $\alpha = e^2/4\pi$ is the fine-structure constant.

e) Consider the elastic scattering $e^-(k_1)\,P(p) \to e^-(k_2)\,P(p')$. In this case the hadronic part is given by an elastic QED-like term as follows

$$W^{\text{elastic}}_{\mu\nu} = L_{qq,\mu\nu} \frac{\mathrm{d}^3 p'}{2p'^0 (2\pi)^3} (2\pi)^4 \delta^{(4)}(p + q - p')\,. \tag{3}$$

Integrate out the momenta p' to show that $W^{\text{elastic}}_{\mu\nu} = L_{qq,\mu\nu} \frac{2\pi x}{Q^2} \delta(1-x)$.

f) (optional) Use Lorentz symmetry, and the fact that $W_{\mu\nu}$ is a conserved current, to show that the most general form for the hadronic tensor in (2) is

$$W^{\mu\nu} \;=\; \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) W_1 + \left(p^\mu - \frac{p\cdot q}{q^2}\, q^\mu\right)\left(p^\nu - \frac{p\cdot q}{q^2}\, q^\nu\right) W_2\,,$$

where $W_{1,2}$ are Lorentz scalar functions.

Exercise 2: DGLAP equations

The splitting functions P_{ij} enter in the evolution of parton distribution functions f_i

$$Q^2 \frac{\partial f_i}{\partial Q^2}(x, Q^2) = \frac{\alpha_s}{2\pi} \int_x^1 \frac{\mathrm{d}z}{z} P_{ij}(z) f_j(\frac{x}{z}, Q^2)\,, \tag{4}$$

which go by the name of DGLAP evolution equations. These integro-differential equations can be simplified using the plus-prescription, defined by

$$\int_0^1 \mathrm{d}x\, f(x) g_+(x) := \int_0^1 \mathrm{d}x\,[f(x) - f(1)] g(x)\,, \tag{5}$$

and the Mellin transform:

$$f_n := \int_0^1 \mathrm{d}x\, x^{n-1} f(x)\,. \tag{6}$$

a) Show that

$$\left(\frac{1+z^2}{1-z}\right)_+ \;=\; \frac{1+z^2}{(1-z)_+} \;+\; \frac{3}{2}\delta(1-z)\,. \tag{7}$$

b) Show that

$$\left(\frac{z}{1-z} + \frac{z(1-z)}{2}\right)_+ = \frac{z}{(1-z)_+} \;+\; \frac{z(1-z)}{2} \;+\; \frac{11}{12}\delta(1-z)\,. \tag{8}$$

c) Apply the Mellin transformation (to both the PDFs f_i and the splitting functions P_{ij}) in order to derive a *simple* differential equation.

d) Perform the Mellin transformation for the quark and gluon splitting functions:

$$P_{qq}(z) \;=\; C_F \left(\frac{1+z^2}{1-z}\right)_+ \tag{9}$$

$$P_{gg}(z) \;=\; 2\,C_A \left(\frac{z}{1-z} + \frac{z(1-z)}{2}\right)_+ \;+\; \frac{1-z}{z} \;+\; \frac{z(1-z)}{2} \tag{10}$$

where C_F and C_A are the $SU(3)$ quadratic Casimirs for the fundamental and adjoint representations, respectively.