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General Relativity
Sheet 13

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Note: In this exercise we will use the convention $\eta_{\mu\nu} = \text{diag}(-,+,+,+)$.

Exercise 1 [Linearized field equations]

In linearized gravity we consider the metric to be of the form $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ where $|h_{\mu\nu}| \ll 1$. This allows us to drop all terms $\mathcal{O}((h_{\mu\nu})^2)$. Furthermore, we drop all derivatives $\mathcal{O}((\partial_\rho h_{\mu\nu})^2)$.

a) Show, that the Einstein field equations reduce to the linearized field equations

$$\Box h_{\mu\nu} + h_{,\mu,\nu} - h^{\rho}{}_{\mu,\rho,\nu} - h^{\rho}{}_{\nu,\rho,\mu} = -16\pi G \left(T_{\mu\nu} - \frac{1}{2} T \eta_{\mu\nu} \right), \quad (1)$$

where $\Box = \partial^\rho \partial_\rho$ is the d'Alembert operator and $h = h^{\mu}{}_{\mu}$ is the trace of the perturbation, by following these steps:

i) Show that the inverse metric is given by $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$ and that $h = \eta^{\mu\nu} h_{\mu\nu}$.

ii) Calculate the Ricci tensor: $R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu} = \partial_\rho \Gamma^{\rho}_{\nu\mu} - \partial_\nu \Gamma^{\rho}_{\rho\mu} + \Gamma^{\rho}_{\rho\lambda}\Gamma^{\lambda}_{\nu\mu} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\rho\mu}$

iii) Use the Einstein field equations: $R_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} \right)$

b) i) Consider a change of coordinates $\tilde{x}^\mu = x^\mu - \xi^\mu(x)$, with $\xi^\mu(x)$ being a vector field and $|\partial_\alpha \xi^\mu| \ll 1$. Show that under this change of coordinates, $h_{\mu\nu}$ changes to

$$\tilde{h}_{\mu\nu} = h_{\mu\nu} + (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu), \quad (2)$$

where all terms of order $\mathcal{O}((\partial\xi)^2)$ have been dropped. This is called a 'gauge transformation': By choosing the four functions ξ_μ, we obtain four constraints on $h_{\mu\nu}$.

ii) Show that, choosing the gauge $h_{\mu\nu}{}^{,\mu} = \frac{1}{2} h_{,\nu}$, the field equations read

$$\Box h_{\mu\nu} = -16\pi G \left(T_{\mu\nu} - \frac{1}{2} T \eta_{\mu\nu} \right). \quad (3)$$

iii) Show that for a static, general source described by $T_{\mu\nu} = \text{diag}\{\rho, 0, 0, 0\}$, the metric $g_{\mu\nu}$ reduces to:

$$ds^2 = -(1 + 2\phi)\, dt^2 + (1 - 2\phi)\left(dx^2 + dy^2 + dz^2 \right), \quad (4)$$

where ϕ is the Newtonian gravitational potential associated to ρ.

c) Consider the trace reversed form of the metric perturbations,

$$\gamma_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h. \quad (5)$$

Show that in the Lorenz gauge, (in which the divergence of $\gamma_{\mu\nu}$ vanishes: $\gamma_{\mu\nu}{}^{,\nu} = 0$), the linearized field equations become

$$\Box \gamma_{\mu\nu} = -16\pi G T_{\mu\nu}. \quad (6)$$

Exercise 2 [The quadrupole radiation formula]

The most general solution for the linearized field equations in the Lorenz gauge (6) is given by

$$\gamma_{\mu\nu}(t,\boldsymbol{x}) = 4G\int d^3y \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|} T_{\mu\nu}\left(t-|\boldsymbol{x}-\boldsymbol{y}|,\boldsymbol{y}\right)\,, \tag{7}$$

with the retarded time $t_r = t - |\boldsymbol{x}-\boldsymbol{y}|$. Show that when the source is isolated, far away and slowly moving this reduces to the well known quadrupole radiation formula

$$\gamma_{ij}(t,\boldsymbol{x}) = \frac{2G}{r}\frac{d^2 I_{ij}}{dt^2}\bigg|_{t_r}\,, \tag{8}$$

where $I_{ij}(t) = \int d^3y\, y^i y^j T^{00}(t,\boldsymbol{y})$ is the quadrupole moment of the source and $t_r = t-r$ is the retarded time. Follow this outline:

i) Put Eq. (7) in the frequency domain by the Fourier transforms

$$\widetilde{\phi}(\omega,x) = \frac{1}{\sqrt{2\pi}}\int dt e^{-i\omega t}\phi(t,x)\,. \tag{9}$$

Change the integration variable from t to $t_r = t - |\boldsymbol{x}-\boldsymbol{y}|$ by multiplying with a suitable exponential and note that $dt = dt_r$.

ii) We are interested in the field far away from the isolated source. Approximate $|\boldsymbol{x}-\boldsymbol{y}| = r$, where r is just the distance to the source. Also forget about the time components of the metric perturbation, as they are related to the spatial components by the Lorenz gauge $\partial_\mu\gamma^{\mu 0} = 0 \Rightarrow \partial_0\gamma^{00} = -\partial_i\gamma^{i0}$.

iii) Use the product rule $\partial_k(y^j\widetilde{T}^{ik}) = y^j\partial_k\widetilde{T}^{ik} + \widetilde{T}^{ij}$ and the conservation law of the Energy momentum tensor in Fourier space $i\omega\widetilde{T}^{0\nu} = -\partial_i\widetilde{T}^{i\nu}$ to relate the ij component to the $i0$ component. Symmetrize the result and use the product rule once more to fully relate the spatial components to the time components.

iv) Transform your equation back to the time domain by applying the inverse Fourier transform

$$\phi(t,x) = \frac{1}{\sqrt{2\pi}}\int d\omega e^{i\omega t}\widetilde{\phi}(\omega,x)\,. \tag{10}$$

Exercise 3 [Gravitational waves from a Binary System]

A binary system with masses m_1 and m_2 is in a circular configuration with radius R. Consider the orbit to be adequately described by Newtonian gravity. We will use this description to compute the leading-order effects of gravitational-wave emission. Don't forget that orbits in a problem of this type are most easily described using the 'reduced system': A body of mass $\mu = m_1 m_2/(m_1+m_2)$ in a circular orbit around a body of mass $M = m_1 + m_2$.

From Ex. 2 you know the solution to the trace reversed linearized Einstein equations

$$\gamma_{ij}(t,\boldsymbol{x}) = \frac{2G}{r}\frac{d^2 I_{ij}}{dt^2}\bigg|_{t_r}\,. \tag{11}$$

The energy loss of such a system is given by

$$P = -\frac{G}{5}\left\langle \frac{d^3 J_{ij}}{dt^3}\frac{d^3 J^{ij}}{dt^3}\right\rangle\bigg|_{t_r}, \tag{12}$$

with $J_{ij} = I_{ij} - \frac{1}{3}\delta_{ij}\delta^{kl}I_{kl}$ being the traceless part of the quadrupole moment and the brackets mean averaging over one orbit.

a) Compute the gravitational-wave tensor γ_{ij} as measured by an observer looking down the angular momentum axis of the system.

b) Compute the rate at which energy is carried away from the system by gravitational waves.

c) By asserting *global* conservation of energy in the following form,

$$\frac{d}{dt}\left(E_{\text{kinetic}} + E_{\text{potential}} + E_{\text{GW}}\right) = 0\,, \tag{13}$$

derive an equation for dR/dt, the rate at which the orbital radius shrinks.

d) Derive the change of the orbital angular frequency ω, caused by the gravitational wave emission. You should find that the masses appear only in the combination $\mu^{3/5}M^{2/5}$, perhaps raised to some power. This combination of masses is known as the 'chirp mass', perhaps raised to some power, since it sets the rate at which the frequency chirps[1].

e) Integrate the $d\omega/dt$ you obtained in part (d), to obtain $\omega(t)$, the time evolution of the binary's orbital frequency. Let T_{coal} (*coalescence time*), be the time at which the inspiral is over, and the frequency goes to infinity.

f) Chickens also chirp. Draw a funky chicken.

[1] chirp = zwitschern