# MMP 2
# Solutions 1

FS 19
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**Exercise 1**

$$\underbrace{(\vec{\nabla}^2 - m^2)}_{L}\phi(\vec{x}) = J(\vec{x})$$

a) Fundamental solution: $LE = \delta \overset{\text{Fourier Space}}{\rightarrow} \widehat{LE} = \hat{\delta} = 1$

$$\begin{aligned}
\widehat{LE} &= \int e^{-i\vec{p}\vec{x}}(\vec{\nabla}^2 - m^2)E(x)d^3x \overset{(1)}{=} \int \vec{\nabla}^2 e^{-i\vec{p}\vec{x}} \cdot E(\vec{x})d^3x - m^2 \int e^{-i\vec{p}\vec{x}}E(\vec{x})d^3x \\
&= -(\vec{p}^2 + m^2)\hat{E}(\vec{p}) = 1 \\
\Rightarrow \hat{E}(\vec{p}) &= \frac{-1}{(\vec{p}^2 + m^2)}
\end{aligned}$$

There are no singularities since $m > 0$

(1) : two times partial integration, boundary terms vanish.

b)

$$\begin{aligned}
F\left[\int f(\vec{y})g(\vec{x}-\vec{y})d\vec{y}\right] &= \iint e^{-i\vec{p}\vec{x}}f(\vec{y})g(\vec{x}-\vec{y})d\vec{x}d\vec{y} \\
&= \int f(\vec{y}) \underbrace{\int e^{-i\vec{p}\vec{x}}g(\vec{x}-\vec{y})d\vec{x}}_{\vec{z}:=\vec{x}-\vec{y};d\vec{x}=d\vec{z};\vec{x}=\vec{z}+\vec{y}} d\vec{y} \\
&= \int f(\vec{y}) \int e^{-i\vec{p}(\vec{z}+\vec{y})}g(\vec{z})d\vec{z}d\vec{y} \\
&= \int f(\vec{y})e^{-i\vec{p}\vec{y}}d\vec{y} \cdot \int e^{-i\vec{p}\vec{z}}g(\vec{z})d\vec{z} \\
&= \hat{f}(\vec{p})\hat{g}(\vec{p})
\end{aligned}$$

c)

$$\begin{aligned}
\phi(\vec{x}) = E * J &= \int E(\vec{x}-\vec{y})J(\vec{y})d\vec{y} \\
\Rightarrow \hat{\phi}(\vec{p}) &= F\left[\int E(\vec{x}-\vec{y})J(\vec{y})d\vec{y}\right] \overset{(b)}{=} \hat{E}(\vec{p})\hat{J}(\vec{p}) \\
\Rightarrow \hat{\phi}(\vec{p}) &= -\frac{1}{\vec{p}^2 + m^2}\int e^{-i\vec{p}\vec{x}}J(\vec{x})d\vec{x}
\end{aligned}$$

**Exercise 2**

$$
\begin{aligned}
z^n &= a \\
\Rightarrow z_k &= \sqrt[n]{|a|} exp\left(\left[i\left(\frac{\alpha + 2\pi k}{n}\right)\right]\right) \\
a &= |a|e^{i\alpha} \\
\Rightarrow |z_k| &= \sqrt[n]{|a|} \\
\varphi_k &= \frac{\alpha + 2\pi k}{n}
\end{aligned}
$$

a) $a = i$ , $n = 4$

$$
\begin{aligned}
&\Rightarrow |a| = 1 \Rightarrow |z_k| = 1 \\
&\Rightarrow \alpha = \frac{\pi}{2} \Rightarrow \varphi_k = \frac{\frac{\pi}{2} + 2\pi k}{4} = \frac{\pi}{8}(1 + 4k) \\
&\Rightarrow \varphi_k = \{\frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}\} \mathrel{\widehat{=}} \{22.5°, 112.5°, 202.5°, 252.5°\}
\end{aligned}
$$

b) $a = -1 - i$ , $n = 2$

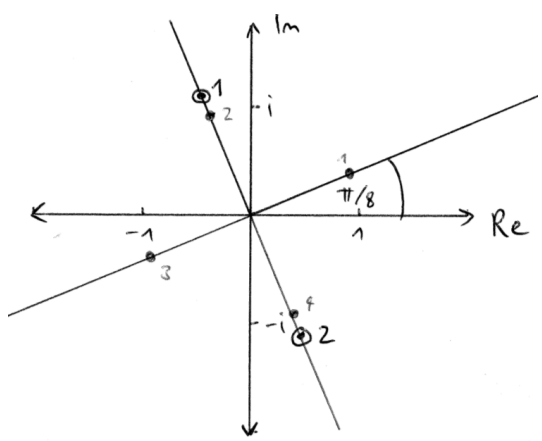
$\Rightarrow |a| = \sqrt{2} \Rightarrow |z_k| = \sqrt[4]{2} \approx 1.19$

$\alpha$ : naively: $\alpha = \arctan \frac{-1}{-1} = \frac{\pi}{4}$

$\Rightarrow$ arctan will return values $\in [-\frac{\pi}{2}, \frac{\pi}{2}]$

But since $\tan(x) = \tan(x + k\pi)$ : $\arctan(y) = x + k\pi$ , you have to check if you've gotten the right result.

$\Rightarrow$ it's *your* responsibility to check your result!




Plot

**Exercise 3**

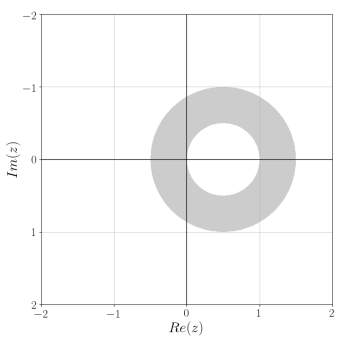
a)

$$\frac{(1+i)^5}{(1-i)^5} = \left(\frac{1+i}{1-i}\right)^5 = \left(\frac{(1+i)^2}{(1-i)(1+i)}\right)^5 = \left(\frac{1+2i-1}{2}\right)^5 = i^5 = (i^2)^2 i = i$$

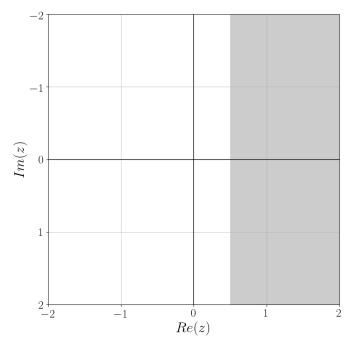
$$(1+i)^n + (1-i)^n = (1+i)^n + \overline{(1+i)^n} = 2Re((1+i)^n) = 2Re(\sqrt{2}^n e^{i\frac{n\pi}{4}}) = \sqrt{2}^{n+2}\cos\left(\frac{n\pi}{4}\right)$$

b) $-\sqrt{3} = \sqrt{3}e^{i\pi}, \quad 4i = 4e^{i\frac{\pi}{2}}, \quad 1+i = \sqrt{2}e^{i\frac{\pi}{4}}$

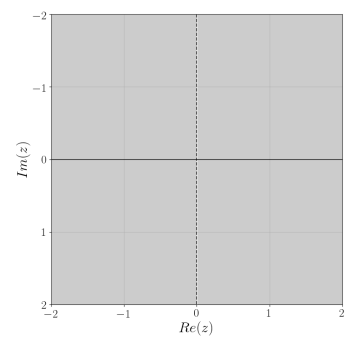
c)




$\{z \in \mathbb{C} | 1 < |2z-1| < 2\}$




$\{z \in \mathbb{C} || z-1| < |z|\}$




$\{z \in \mathbb{C} || Im(z)| < |z|\} = \mathbb{C} \setminus i\mathbb{R}$

**Exercise 4**

- Since the matrix addition is a componentwise addition of real numbers it inherits its abelian group structure, whith the zero matrix as the additive neutral element.
  It remains to show, that $K$ is closed under matrix addition and includes the zero matrix:

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} + \begin{pmatrix} x & -y \\ y & x \end{pmatrix} = \begin{pmatrix} a+x & -(b+y) \\ b+y & a+x \end{pmatrix} \in K,$$

$$x = y = 0 \in \mathbb{R} \quad \Rightarrow \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in K$$

- All matrices in $K \setminus 0$ are invertible:

$$\det \begin{pmatrix} x & -y \\ y & x \end{pmatrix} = x^2 + y^2 \overset{!}{=} 0 \quad \Leftrightarrow \quad x = y = 0$$

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x & -y \\ y & x \end{pmatrix} = \underbrace{\begin{pmatrix} ax - by & -(ay + bx) \\ ay + bx & ax - by \end{pmatrix}}_{\in K} = \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \tag{1}$$

  and the above calculation shows that $K$ is an abelian subgroup of $(\mathbb{R}^{2\times 2}, \cdot)$.

- Matrix addition and multiplication satisfy the distributive rules and thus $K$ is a field.

- Finally consider the map:

$$f : \mathbb{C} \to K, \quad z \mapsto \begin{pmatrix} Re(z) & -Im(z) \\ Im(z) & Re(z) \end{pmatrix}$$

  $f$ is bijective, since $f^{-1} \begin{pmatrix} x & -y \\ y & x \end{pmatrix} = x + iy$ is its inverse. With the following calculations we check, that it is a field homomorphism (preserves the field structure). Let $z = x + iy$ and $w = a + ib \in \mathbb{C}$:

$$f(0_{\mathbb{C}}) = 0_K, \quad f(1_{\mathbb{C}}) = \mathbb{1} = 1_K$$

$$f(z+w) = \begin{pmatrix} x+a & -y-b \\ y+b & x+a \end{pmatrix} = \begin{pmatrix} x & -y \\ y & x \end{pmatrix} + \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = f(z) + f(w)$$

$$f(zw) = f(ax - by + i(bx + ay)) = \begin{pmatrix} ax - by & -(ay + bx) \\ ay + bx & ax - by \end{pmatrix} \overset{(1)}{=} f(z)f(w) \quad \square$$