Exercise 1. *One-loop vacuum polarisation in QED*

1. In massive QED,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\not{D}\psi - m\bar{\psi}\psi\,, \tag{1}$$

consider quantum corrections to the photon polarisation function $i\Pi^{\mu\nu}(q)$. Show that, at one-loop order, $\Pi^{\mu\nu}(q)$ receives contribution from a single diagram. Apply the Feynman rules to verify that the expression of the latter is

$$i\Pi_2^{\mu\nu}(p) = -e^2\mu^{2\epsilon}\,\mathrm{Tr}[\mathbf{1}]\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{N^{\mu\nu}}{(k^2-m^2)((k+p)^2-m^2)}\,, \tag{2}$$

with $\epsilon = (4-d)/2$ and $N^{\mu\nu} = 2k^\mu k^\nu + p^\mu k^\nu + k^\mu p^\nu - g^{\mu\nu}(k^2 + k\cdot p - m^2)$.

2. Show that, by symmetry, one has

$$\begin{aligned}\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{k^\mu}{(k^2-\Delta)^a} &= 0\,,\\ \int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{k^\mu k^\nu}{(k^2-\Delta)^a} &= \frac{1}{d}\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{g^{\mu\nu}k^2}{(k^2-\Delta)^a}\,.\end{aligned} \tag{3}$$

3. Use the Feynman parametrisation to show that

$$i\Pi_2^{\mu\nu}(p) = -e^2\mu^{2\epsilon}\,\mathrm{Tr}[\mathbf{1}]\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{N'^{\mu\nu}}{(k^2-\Delta)^2}\,, \tag{4}$$

with $\Delta = m^2 - x(1-x)p^2$ and

$$N'^{\mu\nu} = g^{\mu\nu}k^2\left(\frac{2}{d}-1\right) - 2x(1-x)p^\mu p^\nu + g^{\mu\nu}(x(1-x)p^2+m^2)\,. \tag{5}$$

Finally, use the result of the Exercise Sheet 1 to integrate over the loop momentum and obtain

$$i\Pi_2^{\mu\nu}(p) = -i(p^2g^{\mu\nu} - p^\mu p^\nu)\Pi_2(p^2)\,, \tag{6}$$

with

$$\Pi_2(p^2) = \frac{2e^2\mu^{2\epsilon}}{(4\pi)^{d/2}}\,\mathrm{Tr}[\mathbf{1}]\,\Gamma\left(2-\frac{d}{2}\right)\int_0^1\mathrm{d}x\,x(1-x)\Delta^{d/2-2}\,. \tag{7}$$

Argue that this result is consistent with gauge invariance.

In the case of massless electron, compute the integral over the Feynman parameter and get

$$\Pi_2(p^2)|_{m=0} = \frac{2e^2\mu^{2\epsilon}}{(4\pi)^{d/2}}\,\mathrm{Tr}[\mathbf{1}](-p^2)^{d/2-2}\,\frac{\Gamma\left(2-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}\right)^2}{\Gamma(d)}\,. \tag{8}$$

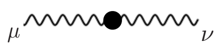
In the $m \neq 0$ case, expand $i\Pi_2(p^2)$ around $\epsilon = 0$ ($\mathrm{Tr}[\mathbf{1}] = 4$) and verify that the vacuum polarisation has a single pole in four dimensions, whose residue corresponds to the one of the massless calculation (8),

$$\Pi_2(p^2) = \frac{e^2}{2\pi^2}\int_0^1\mathrm{d}x\,x(1-x)\left(\frac{1}{\epsilon} - \gamma_E + \log 4\pi - \log\frac{\Delta}{\mu^2}\right) + \mathcal{O}(\epsilon)\,. \tag{9}$$

4. In the renormalised QED Lagrangian, $\mathcal{L}_R = \mathcal{L} + \mathcal{L}_{\text{c.t.}}$,

$$\mathcal{L}_{\text{c.t.}} = -\frac{1}{4}\delta_3 F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\left(\delta_2 \not{\partial} + ie_R\delta_1 \not{A}\right)\psi + (\delta_2+\delta_m)m_R\bar{\psi}\psi\,, \tag{10}$$

   the photon wave function renormalisation constant δ_3 introduces a new coupling,



$$= -i\delta_3(p^2 g^{\mu\nu} - p^\mu p^\nu)\,, \tag{11}$$

   which contributes to $i\Pi^{\mu\nu}$. In the (modified) *minimal subtraction* scheme ($\overline{\text{MS}}$), the counterterms are chosen in order to remove, besides the UV divergent terms of Green's functions, also the finite terms proportional to γ_E and $\log 4\pi$, which are produced by dimensional regularisation. Compute $\delta_3^{\overline{\text{MS}}}$.

5. Alternatively, in the *on-shell* renormalisation scheme (OS), δ_i are fixed in such a way that the parameters of the renormalised Lagrangian correspond to the physical ones. In the case of the photon wave function, this amounts to require that the renormalised propagator $iG^{\mu\nu}$ has a pole at $p^2=0$ with residue $-i$.

   By starting from the bare photon propagator in Lorentz gauge,

$$iG^{\mu\nu}_{\text{bare}} = -i\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)\frac{1}{p^2}\,, \tag{12}$$

   compute the renormalised propagator $iG^{\mu\nu}$ up to $\mathcal{O}(e_R^4)$ and determine δ_3^{OS}.

   Verify that, conversely to the $\overline{\text{MS}}$ scheme, in the on-shell scheme the renormalised propagator becomes independent of the subtraction point μ.