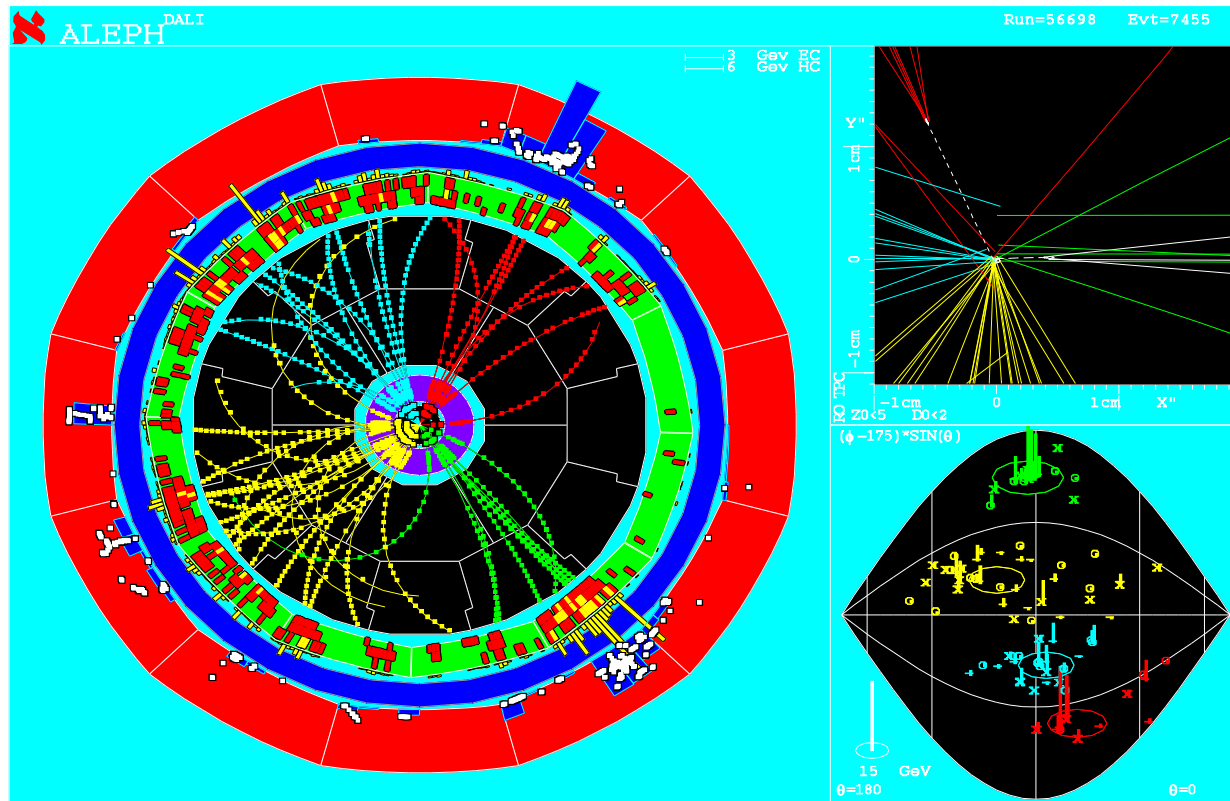


Particle Physics

Handout from Prof. Mark Thomson's lectures
Adapted to UZH by Prof. Canelli and Prof. Serra



Handout 1 : Introduction

UZH Particle Physics Courses

“Particle and Nuclear Physics”
Prof Kilminster/Serra

Introductory course



“Particle Physics”
Prof Canelli/Serra

Covering most Standard Model physics, both experiment and underlying theory

Course Synopsis

Based on the book “Modern Particle Physics”

Chapters 1-3: Introduction, Decay Rates and Cross Sections

Chapter 4: The Dirac Equation and Spin

Chapter 5: Interaction by Particle Exchange

Chapter 6: Electron – Positron Annihilation

Chapter 7: Electron – Proton Scattering

Chapter 8: Deep Inelastic Scattering

Chapter 9: Symmetries and the Quark Model

Chapter 10: QCD and Colour

Chapter 11: V-A and the Weak Interaction

Chapter 12: Leptonic Weak Interactions

Chapter 13: Neutrinos and Neutrino Oscillations

Chapter 14: The CKM Matrix and CP Violation

Chapter 15: Electroweak Unification and the W and Z Bosons

Chapter 16: Tests of the Standard Model

Chapters 17-18: The Higgs Boson and Beyond



- ★ Will concentrate on the modern view of particle physics with the emphasis on how **theoretical concepts** relate to recent **experimental measurements**
- ★ Aim: by the end of the course you should have a good understanding of both aspects of particle physics

Preliminaries

Web-page: <http://www.physik.uzh.ch/lectures/ktp/>

- Homeworks and solutions will be posted here about every 3 classes
- Instructor schedule

Grade:

- 40% homework, 60% final exam

Format of Lectures/Handouts:

- Will be taking M. Thomson Handouts as guideline

Books:

★ **Main book:**

“Modern Particle Physics”, Mark Thomson (Cambridge)

★ **The handouts are fairly complete, however there a number of decent books:**

- “Particle Physics”, Martin and Shaw (Wiley): fairly basic but good.
- “Introductory High Energy Physics”, Perkins (Cambridge): slightly below level of the course but well written.
- “Introduction to Elementary Physics”, Griffiths (Wiley): about right level but doesn't cover the more recent material.
- “Quarks and Leptons”, Halzen & Martin (Wiley): good graduate level textbook (slightly above level of this course).

★ Before we start in earnest, a few words on units/notation and a very brief recap of KTI

Preliminaries: Natural Units

- **S.I. UNITS:** kg m s are a natural choice for “everyday” objects
e.g. $M \sim 250$ kg
- not very natural in particle physics
- instead use **Natural Units** based on the language of particle physics
 - From Quantum Mechanics - the unit of action : \hbar
 - From Relativity - the speed of light: c
 - From Particle Physics - unit of energy: **GeV** (1 GeV \sim proton rest mass energy)

★ Units become (i.e. with the correct dimensions):

Energy	GeV	Time	$(\text{GeV}/\hbar)^{-1}$
Momentum	GeV/c	Length	$(\text{GeV}/\hbar c)^{-1}$
Mass	GeV/c^2	Area	$(\text{GeV}/\hbar c)^{-2}$

★ Simplify algebra by setting: $\hbar = c = 1$

• Now all quantities expressed in powers of **GeV**

Energy	GeV	Time	GeV^{-1}
Momentum	GeV	Length	GeV^{-1}
Mass	GeV	Area	GeV^{-2}

To convert back to S.I. units,
need to restore missing factors
of \hbar and c

Preliminaries: Heaviside-Lorentz Units

- **Electron charge** defined by Force equation: $F = \frac{e^2}{4\pi\epsilon_0 r^2}$

- In Heaviside-Lorentz units set $\epsilon_0 = 1$

and $F \rightarrow \frac{e^2}{4\pi r^2}$ **NOW: electric charge** $[FL^2]^{\frac{1}{2}} = [EL]^{\frac{1}{2}} = [\hbar c]^{\frac{1}{2}}$
has dimensions

- **Since** $c = (\epsilon_0\mu_0)^{-\frac{1}{2}} = 1 \rightarrow \mu_0 = 1$

$$\hbar = c = \epsilon_0 = \mu_0 = 1$$

★ Unless otherwise stated, Natural Units are used throughout these handouts, $E^2 = p^2 + m^2$, $\vec{p} = \vec{k}$, etc.

Review of The Standard Model

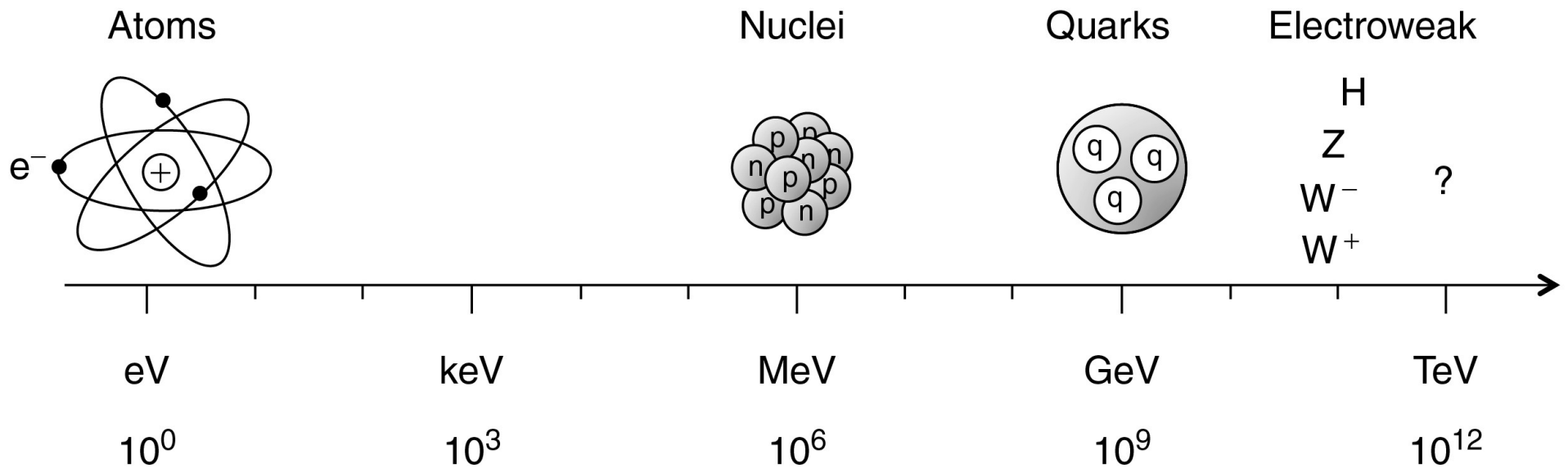
Particle Physics is the study of:

- ★ **MATTER:** the fundamental constituents of the universe
- the elementary particles
- ★ **FORCE:** the fundamental forces of nature, i.e. the interactions
between the elementary particles

Try to categorise the **PARTICLES** and **FORCES** in as simple and fundamental manner possible

- ★ Current understanding embodied in the **STANDARD MODEL:**
 - **Forces** between particles due to exchange of particles
 - Consistent with **all** current experimental data !
 - But it is just a “**model**” with many unpredicted parameters, e.g. particle masses.
 - As such it is not the ultimate theory (if such a thing exists), there are many mysteries.

Review of The Standard Model



★ **Almost all commonly encountered physical phenomenon can be described in terms of electron, electron neutrino, proton, neutron, interacting by the electromagnetic, strong and weak forces**

★ **At higher scale further structure is observed at higher energy scales**

Matter in the Standard Model

- ★ In the Standard Model the fundamental “matter” is described by **point-like spin-1/2 fermions**

	LEPTONS			QUARKS		
		q	m/GeV		q	m/GeV
First Generation	e^-	-1	0.0005	d	-1/3	0.3
	ν_1	0	≈ 0	u	+2/3	0.3
Second Generation	μ^-	-1	0.106	s	-1/3	0.5
	ν_2	0	≈ 0	c	+2/3	1.5
Third Generation	τ^-	-1	1.77	b	-1/3	4.5
	ν_3	0	≈ 0	t	+2/3	175

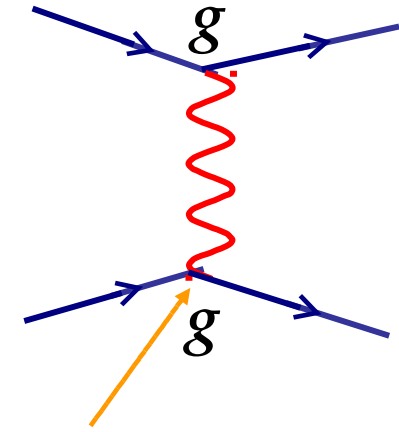
The masses quoted for the quarks are the “constituent masses”, i.e. the effective masses for quarks confined in a bound state

- In the SM there are **three generations** – the particles in each generation are copies of each other differing **only** in mass. (not understood why three)
- The neutrinos are much lighter than all other particles (e.g. ν_1 has $m < 3$ eV) – we now know that neutrinos have non-zero mass (don’t understand why so small)
- **Dirac equation** describes the dynamics of the 12 particles and their anti-particles

Forces in the Standard Model

★ Forces mediated by the exchange of **spin-1 Gauge Bosons**

Force	Boson(s)	J^P	m/GeV
EM (QED)	Photon γ	1^-	0
Weak	W^\pm / Z	1^-	80 / 91
Strong (QCD)	8 Gluons g	1^-	0
Gravity (?)	Graviton?	2^+	0



- Fundamental interaction strength is given by charge g .
- Related to the dimensionless coupling “constant” α

e.g. QED
$$g_{em} = e = \sqrt{4\pi\alpha\epsilon_0\hbar c}$$

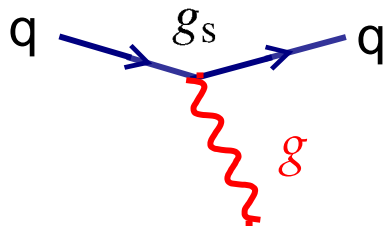
- ★ In Natural Units
$$g = \sqrt{4\pi\alpha}$$
 (both g and α are dimensionless, but g contains a “hidden” $\hbar c$)

- ★ Convenient to express couplings in terms of α which, being genuinely dimensionless does not depend on the system of units (this is not true for the numerical value for e)

Standard Model Vertices

- ★ Interaction of **gauge bosons** with **fermions** described by SM vertices
- ★ Properties of the **gauge bosons** and **nature of the interaction** between the bosons and fermions determine the properties of the interaction

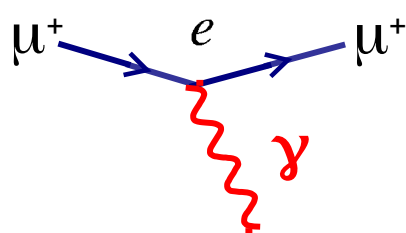
STRONG



Only quarks
Never changes flavour

$$\alpha_s \sim 1$$

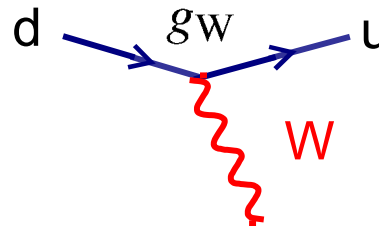
EM



All charged fermions
Never changes flavour

$$\alpha \simeq 1/137$$

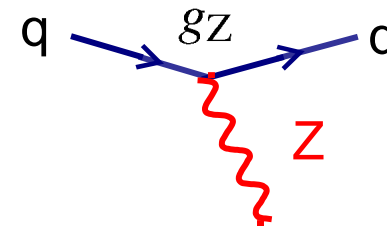
WEAK CC



All fermions
Always changes flavour

$$\alpha_{W/Z} \sim 1/40$$

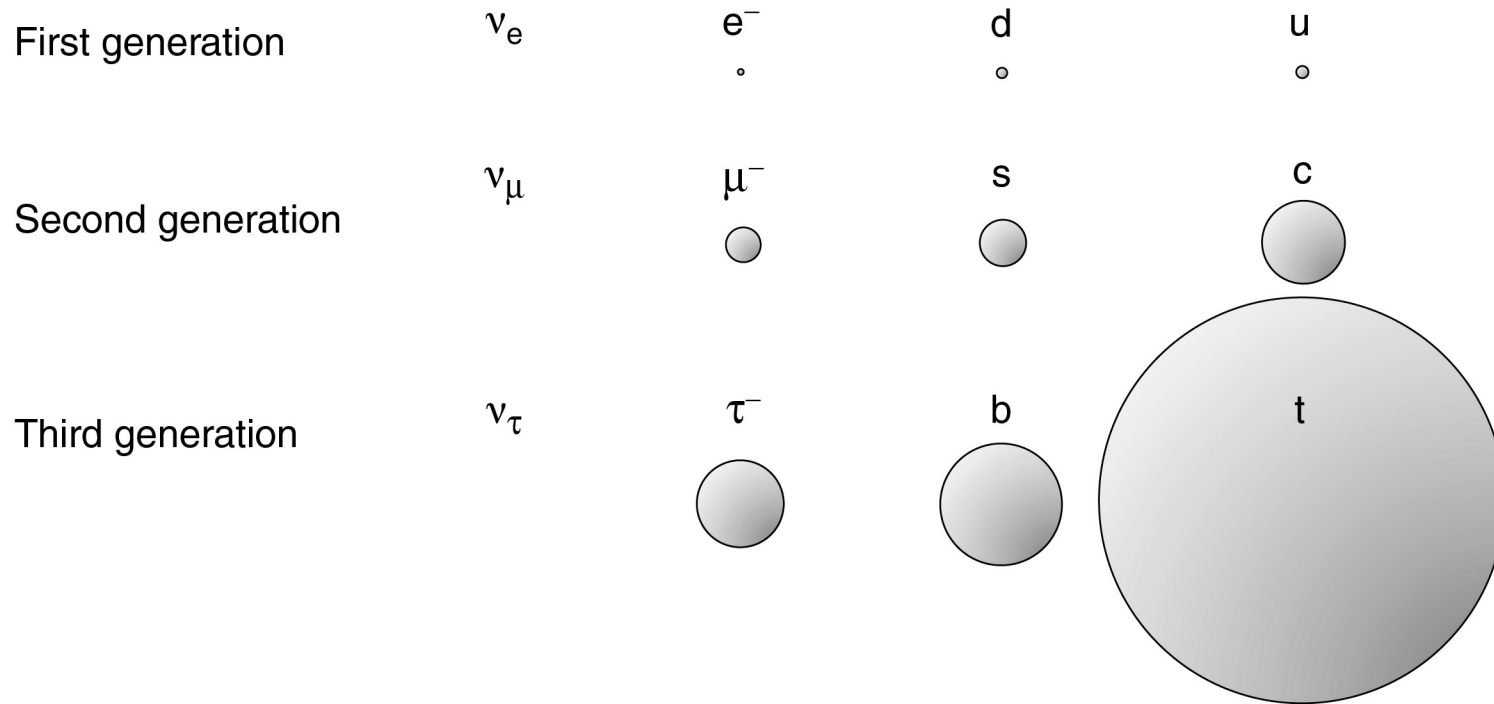
WEAK NC



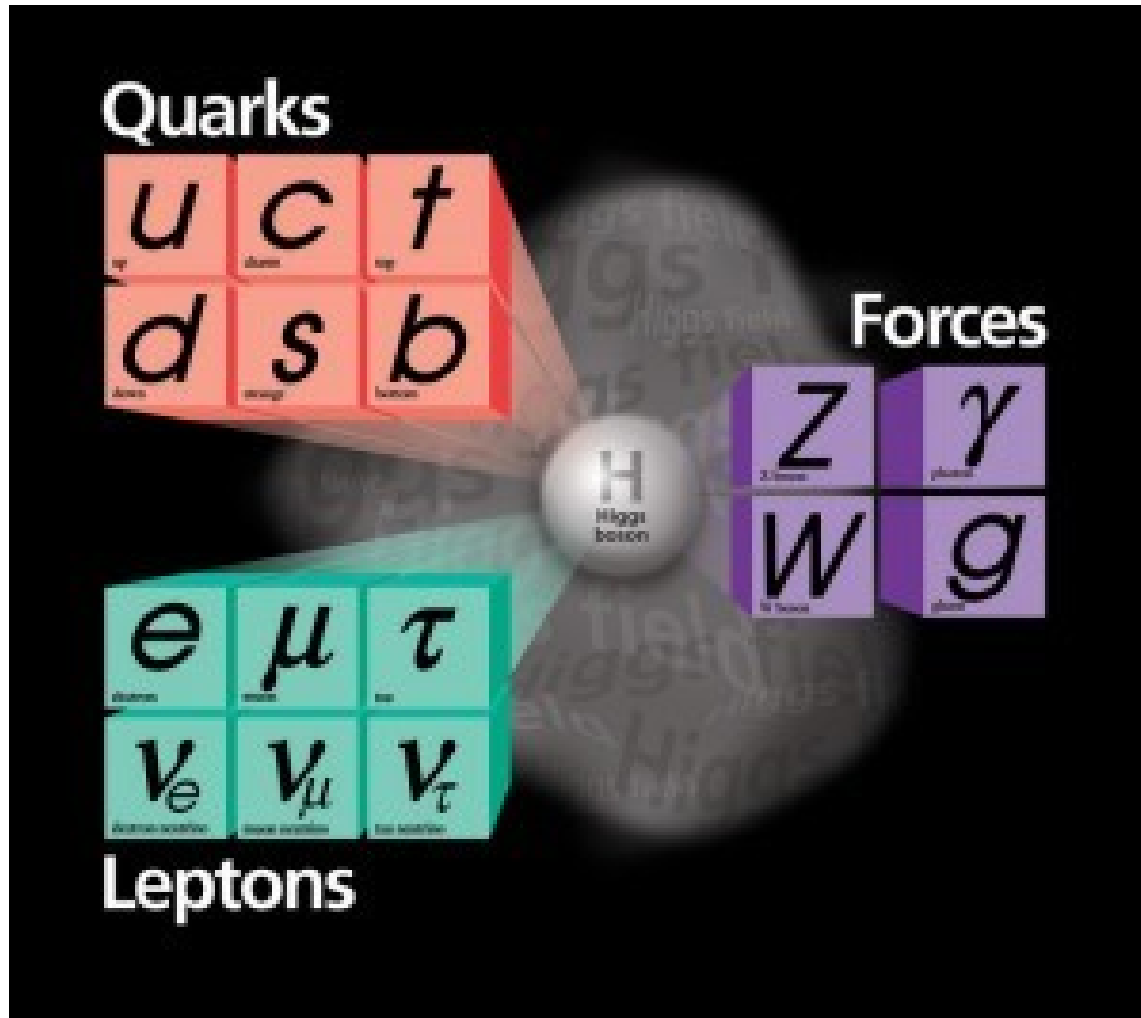
All fermions
Never changes flavour

$$\alpha_{W/Z} \sim 1/40$$

Masses in the Standard Model



The Higgs boson



- ★ Only fundamental scalar particle with **spin-0** Gauge Bosons

- ★ Provides a mechanism by which all other particles acquire mass

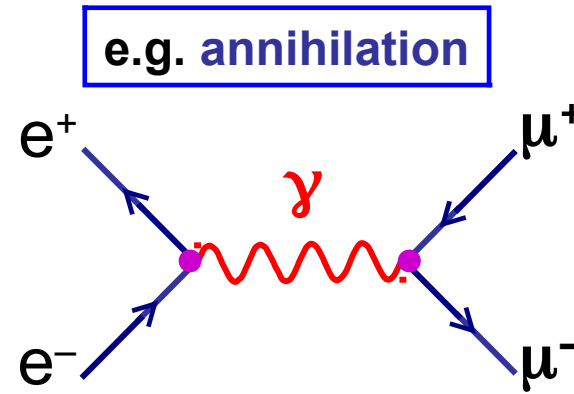
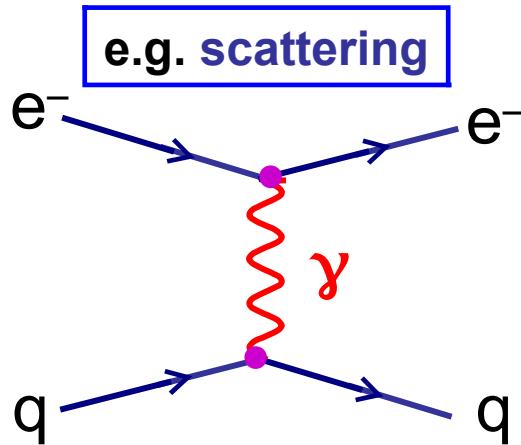
- ★ In QFT the Higgs boson can be thought of an excitation of the Higgs field

- ★ The Higgs field is believed to have a non-zero vacuum expectation value

- ★ Interactions of the initially massless particles with this Higgs field that gives them their masses

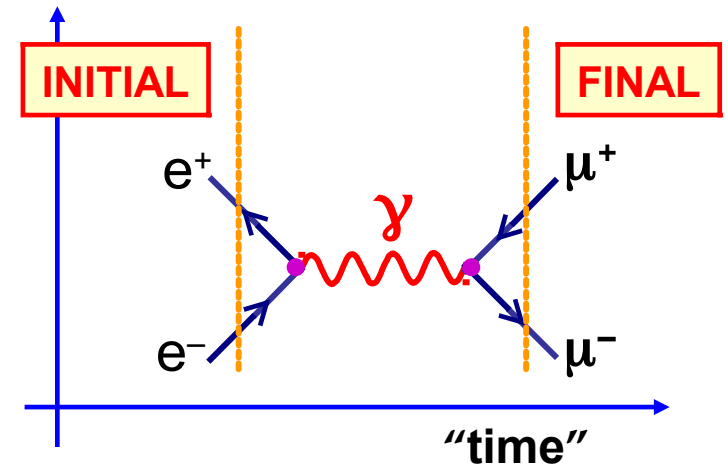
Feynman Diagrams

★ Particle interactions described in terms of Feynman diagrams



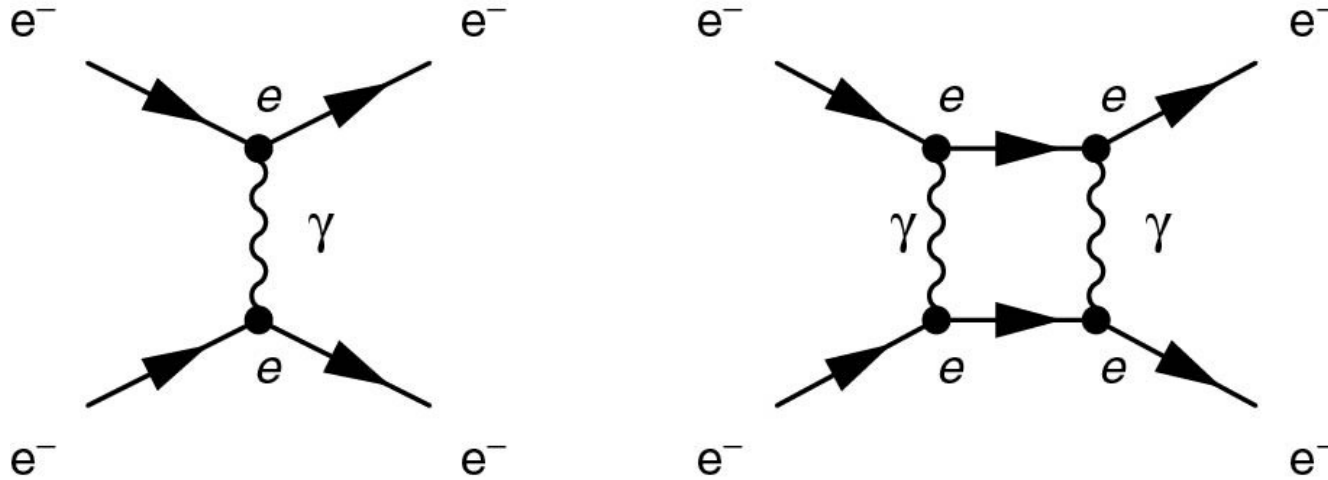
★ IMPORTANT POINTS TO REMEMBER:

- “time” runs from left – right, **only** in sense that:
 - ◆ LHS of diagram is initial state
 - ◆ RHS of diagram is final state
 - ◆ Middle is “how it happened”
- anti-particle arrows in -ve “time” direction
- Energy, momentum, angular momentum, etc. conserved at **all interaction vertices**
- All intermediate particles are “virtual”
i.e. $E^2 \neq |\vec{p}|^2 + m^2$ (handout 3)



Feynman Diagrams

- ★ For a given process many diagrams can be drawn



- ★ Because of the coupling strength diagrams with more vertices are suppressed
- ★ Only the lowest-order term is significant

Special Relativity and 4-Vector Notation

- Will use 4-vector notation with p^0 as the time-like component, e.g.

$$p^\mu = \{E, \vec{p}\} = \{E, p_x, p_y, p_z\} \quad (\text{contravariant})$$

$$p_\mu = g_{\mu\nu} p^\mu = \{E, -\vec{p}\} = \{E, -p_x, -p_y, -p_z\} \quad (\text{covariant})$$

with

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- In particle physics, usually deal with relativistic particles. **Require all** calculations to be **Lorentz Invariant**. **L.I.** quantities formed from 4-vector scalar products, e.g.

$$p^\mu p_\mu = E^2 - p^2 = m^2 \quad \text{Invariant mass}$$

$$x^\mu p_\mu = Et - \vec{p} \cdot \vec{r} \quad \text{Phase}$$

- A few words on NOTATION

Four vectors written as either: p^μ or p

Four vector scalar product: $p^\mu q_\mu$ or $p \cdot q$

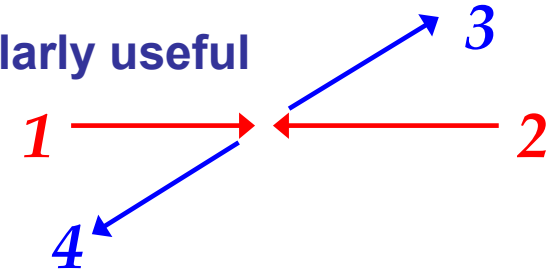
Three vectors written as: \vec{p}

Quantities evaluated in the centre of mass frame: \vec{p}^* , p^* etc.

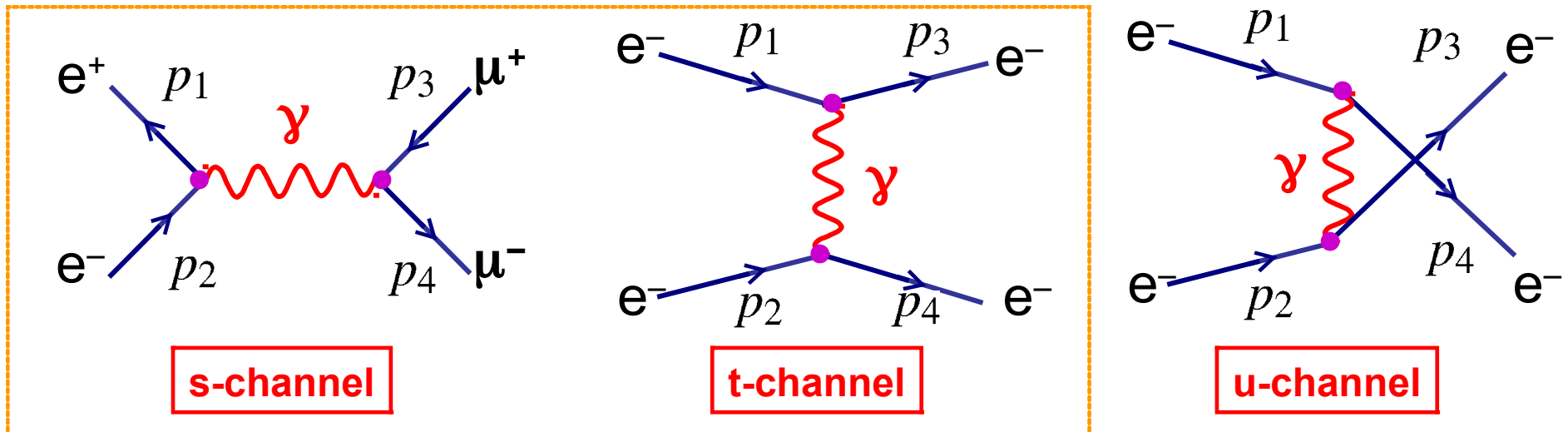
Mandelstam s, t and u

★ In particle scattering/annihilation there are three particularly useful **Lorentz Invariant** quantities: **s, t** and **u**

★ Consider the scattering process $1 + 2 \rightarrow 3 + 4$



★ (Simple) Feynman diagrams can be categorised according to the four-momentum of the exchanged particle



• Can define **three** kinematic variables: **s, t** and **u** from the following four vector scalar products (squared four-momentum of exchanged particle)

$$s = (p_1 + p_2)^2, \quad t = (p_1 - p_3)^2, \quad u = (p_1 - p_4)^2$$

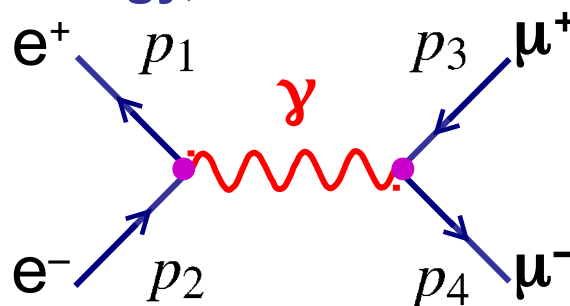
Example: Mandelstam s, t and u

$$s = (p_1 + p_2)^2, \quad t = (p_1 - p_3)^2, \quad u = (p_1 - p_4)^2$$

Note: $s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2$

(Problem 2.12)

★ e.g. Centre-of-mass energy, s :



$$s = (p_1 + p_2)^2 = (E_1 + E_2)^2 - (\vec{p}_1 + \vec{p}_2)^2$$

- This is a scalar product of two four-vectors \rightarrow Lorentz Invariant
- Since this is a **L.I.** quantity, can evaluate in **any** frame. Choose the most convenient, i.e. the centre-of-mass frame:

$$p_1^* = (E_1^*, \vec{p}^*) \quad p_2^* = (E_2^*, -\vec{p}^*)$$

\rightarrow $s = (E_1^* + E_2^*)^2$

★ Hence \sqrt{s} is the total energy of collision in the centre-of-mass frame

From Feynman diagrams to Physics

Particle Physics = Precision Physics

- ★ Particle physics is about building fundamental theories and testing their predictions against precise experimental data
 - Dealing with fundamental particles and can make **very precise theoretical predictions** – not complicated by dealing with many-body systems
 - Many beautiful experimental measurements
 - precise theoretical predictions challenged by precise measurements
 - For all its flaws, the Standard Model describes all experimental data !
This is a **(the?) remarkable achievement of late 20th century physics.**

Requires understanding of theory and experimental data

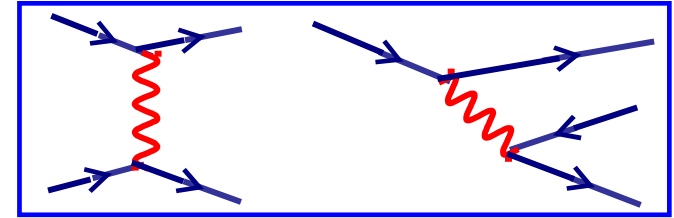
- ★ Develop the methodology for calculation of decay rates and interaction cross sections for particle annihilation and scattering processes
 - ♦ will use Feynman diagrams and associated Feynman rules to perform calculations for many processes
 - ♦ hopefully gain a fairly deep understanding of the Standard Model and how it explains all current data

Before we can start, need calculations for:

- **Interaction cross sections;**
- **Particle decay rates;**

Cross Sections and Decay Rates

- In particle physics we are mainly concerned with particle interactions and decays, i.e. transitions between states



- these are the experimental observables of particle physics

- Calculate transition rates from Fermi's Golden Rule (non-relativistic QM)

$$\Gamma_{fi} = 2\pi |T_{fi}|^2 \rho(E_f)$$

Γ_{fi} is number of transitions per unit time from initial state $|i\rangle$ to final state $\langle f|$ – **not Lorentz Invariant!**

T_{fi} is Transition Matrix Element

$$T_{fi} = \langle f|\hat{H}|i\rangle + \sum_{j \neq i} \frac{\langle f|\hat{H}|j\rangle \langle j|\hat{H}|i\rangle}{E_i - E_j} + \dots$$

\hat{H} is the perturbing Hamiltonian (small)

$\rho(E_f)$ is density of final states

- ★ Rates depend on **MATRIX ELEMENT** and **DENSITY OF STATES**

the ME contains the fundamental particle physics

just kinematics

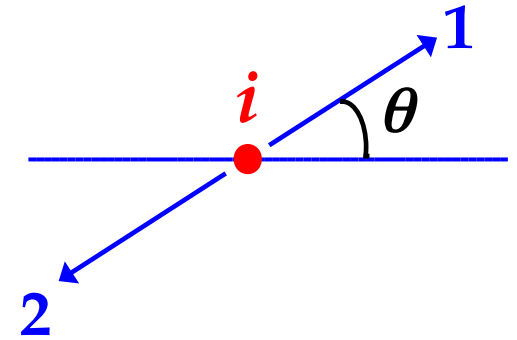
Particle Decay Rates

- Consider the two-body decay

$$i \rightarrow 1 + 2$$

- Want to calculate the decay rate in first order perturbation theory using plane-wave descriptions of the initial and final particles (Born approximation) :

$$\begin{aligned}\psi_1 &= N e^{i(\vec{p}\cdot\vec{r}-Et)} \\ &= N e^{-ip\cdot x}\end{aligned}\quad (\vec{k}\cdot\vec{r} = \vec{p}\cdot\vec{r} \text{ as } \hbar = 1)$$



where N is the normalisation and $p\cdot x = p^\mu x_\mu$

For decay rate calculation need to know:

- Wave-function normalisation
- Expression for the density of states
- Transition matrix element from perturbation theory

All in a Lorentz Invariant form

★ Wave-function normalisation:

- Non-relativistic: usually normalised to one particle in a cube of side a

$$\int \psi \psi^* dV = N^2 a^3 = 1 \Rightarrow N^2 = 1/a^3$$

Non-relativistic Phase Space (revision)

- **Density of states**
- **Wave-function vanishing at box boundaries**
 → **quantised particle momenta:**

$$p_x = \frac{2\pi n_x}{a}; \quad p_y = \frac{2\pi n_y}{a}; \quad p_z = \frac{2\pi n_z}{a}$$

- **Volume of single state in momentum space:**

$$\left(\frac{2\pi}{a}\right)^3 = \frac{(2\pi)^3}{V}$$

- **Normalising to one particle/unit volume gives number of states in element:** $d^3\vec{p} = dp_x dp_y dp_z$

$$dn = \frac{d^3\vec{p}}{(2\pi)^3} \times \frac{1}{V} = \frac{d^3\vec{p}}{(2\pi)^3}$$

- **Therefore density of states in Golden rule:**

$$\rho(E_f) = \left| \frac{dn}{dE} \right|_{E_f} = \left| \frac{dn}{d|\vec{p}|} \frac{d|\vec{p}|}{dE} \right|_{E_f}$$

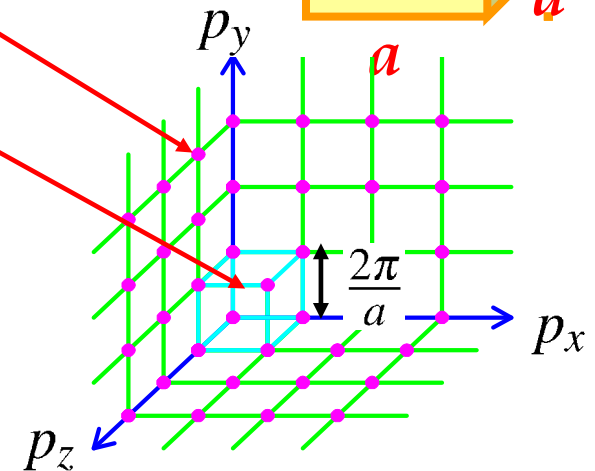
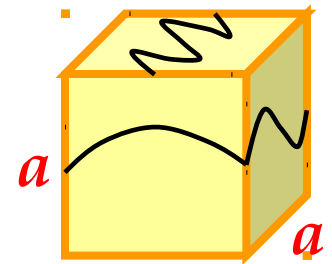
- **Integrating over an elemental shell in momentum-space gives**

$$(d^3\vec{p} = 4\pi p^2 dp)$$

$$\rho(E_f) = \frac{4\pi p^2}{(2\pi)^3} \times \frac{1}{\beta}$$

Apply boundary conditions

$$\vec{p} = \hbar\vec{k}$$



with

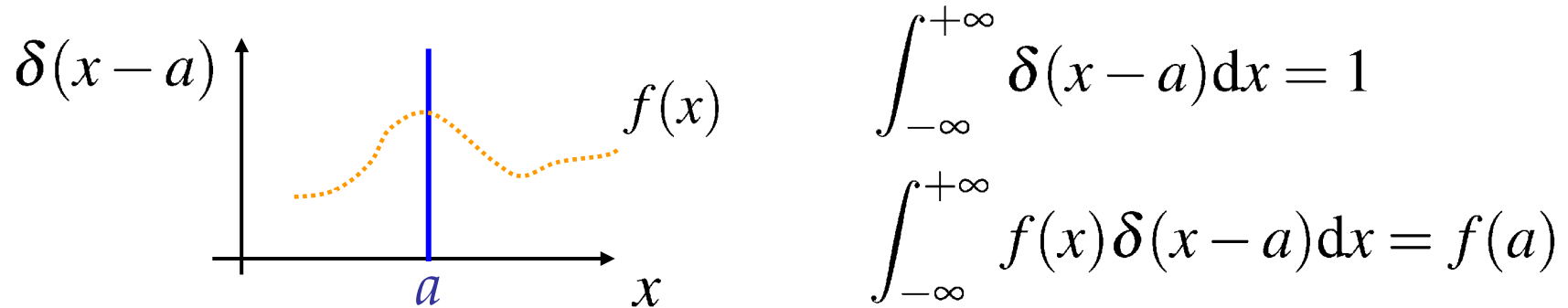
$$E^2 = p^2 + m^2$$

$$2E dE = 2p dp$$

$$\frac{dp}{dE} = \frac{E}{p} = \frac{1}{\beta}$$

Dirac δ Function

- In the relativistic formulation of decay rates and cross sections we will make use of the Dirac δ function: **“infinitely narrow spike of unit area”**



- Any function with the above properties can represent $\delta(x)$

e.g.
$$\delta(x) = \lim_{\sigma \rightarrow 0} \frac{1}{\sqrt{2\pi\sigma}} e^{-\left(\frac{x^2}{2\sigma^2}\right)}$$
 (an infinitesimally narrow Gaussian)

- In relativistic quantum mechanics delta functions prove extremely useful for integrals over phase space, e.g. in the decay $a \rightarrow 1 + 2$

$$\int \dots \delta(E_a - E_1 - E_2) dE \quad \text{and} \quad \int \dots \delta^3(\vec{p}_a - \vec{p}_1 - \vec{p}_2) d^3\vec{p}$$

express energy and momentum conservation

Dirac δ Function

★ We will soon need an expression for the **delta function of a function** $\delta(f(x))$

- Start from the definition of a delta function

$$\int_{y_1}^{y_2} \delta(y) dy = \begin{cases} 1 & \text{if } y_1 < 0 < y_2 \\ 0 & \text{otherwise} \end{cases}$$

- Now express in terms of $y = f(x)$ where $f(x_0) = 0$ and then change variables

$$\int_{x_1}^{x_2} \delta(f(x)) \frac{df}{dx} dx = \begin{cases} 1 & \text{if } x_1 < x_0 < x_2 \\ 0 & \text{otherwise} \end{cases}$$

- From properties of the delta function (i.e. here only non-zero at x_0)

$$\left| \frac{df}{dx} \right|_{x_0} \int_{x_1}^{x_2} \delta(f(x)) dx = \begin{cases} 1 & \text{if } x_1 < x_0 < x_2 \\ 0 & \text{otherwise} \end{cases}$$

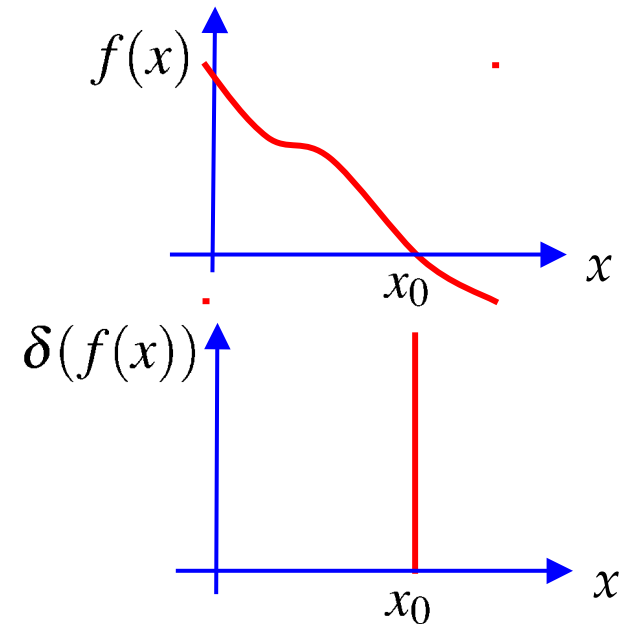
- Rearranging and expressing the RHS as a delta function

$$\int_{x_1}^{x_2} \delta(f(x)) dx = \frac{1}{\left| df/dx \right|_{x_0}} \int_{x_1}^{x_2} \delta(x - x_0) dx$$



$$\delta(f(x)) = \left| \frac{df}{dx} \right|_{x_0}^{-1} \delta(x - x_0)$$

(1)



The Golden Rule revisited

$$\Gamma_{fi} = 2\pi |T_{fi}|^2 \rho(E_f)$$

- Rewrite the expression for density of states using a delta-function

$$\rho(E_f) = \left. \frac{dn}{dE} \right|_{E_f} = \int \frac{dn}{dE} \delta(E - E_i) dE \quad \text{since } E_f = E_i$$

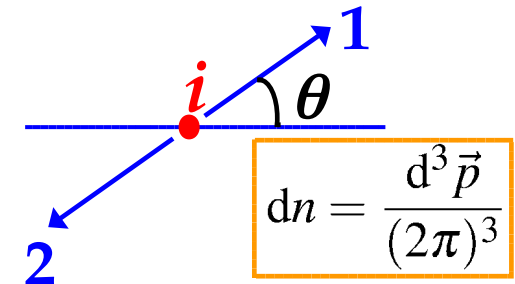
Note : integrating over all final state energies but energy conservation now taken into account explicitly by delta function

- Hence the golden rule becomes: $\Gamma_{fi} = 2\pi \int |T_{fi}|^2 \delta(E_i - E) dn$

the integral is over all “allowed” final states of **any energy**

- For dn in a two-body decay, only need to consider one particle : **mom. conservation** fixes the other

$$\Gamma_{fi} = 2\pi \int |T_{fi}|^2 \delta(E_i - E_1 - E_2) \frac{d^3 \vec{p}_1}{(2\pi)^3}$$

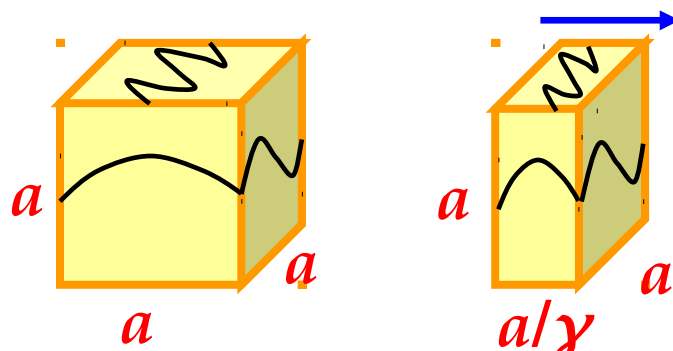


- However, can include momentum conservation explicitly by integrating over the momenta of **both** particles and using another δ -fn

$$\Gamma_{fi} = (2\pi)^4 \int |T_{fi}|^2 \underbrace{\delta(E_i - E_1 - E_2)}_{\text{Energy cons.}} \underbrace{\delta^3(\vec{p}_i - \vec{p}_1 - \vec{p}_2)}_{\text{Mom. cons.}} \underbrace{\frac{d^3 \vec{p}_1}{(2\pi)^3} \frac{d^3 \vec{p}_2}{(2\pi)^3}}_{\text{Density of states}}$$

Lorentz Invariant Phase Space

- In non-relativistic QM normalise to one particle/unit volume: $\int \psi^* \psi dV = 1$
- When considering relativistic effects, volume **contracts** by $\gamma = E/m$



- Particle density therefore increases by $\gamma = E/m$
 - ★ Conclude that a relativistic invariant wave-function normalisation needs to be proportional to E particles per unit volume
- Usual convention: **Normalise to $2E$ particles/unit volume** $\int \psi'^* \psi' dV = 2E$
- Previously used ψ normalised to 1 particle per unit volume $\int \psi^* \psi dV = 1$
- Hence $\psi' = (2E)^{1/2} \psi$ is normalised to $2E$ per unit volume
- **Define Lorentz Invariant Matrix Element**, M_{fi} , in terms of the wave-functions normalised to $2E$ particles per unit volume

$$M_{fi} = \langle \psi'_1 \cdot \psi'_2 \dots | \hat{H} | \dots \psi'_{n-1} \psi'_n \rangle = (2E_1 \cdot 2E_2 \cdot 2E_3 \dots 2E_n)^{1/2} T_{fi}$$

The Golden Rule revisited – Cont.

- For the two body decay

$$i \rightarrow 1 + 2$$

$$\begin{aligned} M_{fi} &= \langle \psi'_1 \psi'_2 | \hat{H}' | \psi_i \rangle \\ &= (2E_i \cdot 2E_1 \cdot 2E_2)^{1/2} \langle \psi_1 \psi_2 | \hat{H}' | \psi_i \rangle \\ &= (2E_i \cdot 2E_1 \cdot 2E_2)^{1/2} T_{fi} \end{aligned}$$

- ★ Now expressing T_{fi} in terms of M_{fi} gives

$$\Gamma_{fi} = \frac{(2\pi)^4}{2E_i} \int |M_{fi}|^2 \delta(E_i - E_1 - E_2) \delta^3(\vec{p}_a - \vec{p}_1 - \vec{p}_2) \frac{d^3 \vec{p}_1}{(2\pi)^3 2E_1} \frac{d^3 \vec{p}_2}{(2\pi)^3 2E_2}$$

Note:

- M_{fi} uses relativistically normalised wave-functions. It is **Lorentz Invariant**
- $\frac{d^3 \vec{p}}{(2\pi)^3 2E}$ is the **Lorentz Invariant Phase Space** for each final state particle
the factor of $\frac{1}{2E}$ arises from the wave-function normalisation (prove this)
- This form of Γ_{fi} is simply a rearrangement of the original equation **but the integral is now frame independent** (i.e. **L.I.**)
- Γ_{fi} is inversely proportional to E_i , the energy of the decaying particle. This is exactly what one would expect from time dilation ($E_i = \gamma m$).
- Energy and momentum conservation in the delta functions

Two Body Decay

$$\Gamma_{fi} = \frac{(2\pi)^4}{2E_i} \int |M_{fi}|^2 \delta(E_i - E_1 - E_2) \delta^3(\vec{p}_i - \vec{p}_1 - \vec{p}_2) \frac{d^3\vec{p}_1}{(2\pi)^3 2E_1} \frac{d^3\vec{p}_2}{(2\pi)^3 2E_2}$$

★ Because the **integral** is Lorentz invariant (i.e. frame independent) it can be evaluated in any frame we choose. The C.o.M. frame is most convenient

• In the C.o.M. frame $E_i = m_i$ and $\vec{p}_i = 0 \Rightarrow$

$$\Gamma_{fi} = \frac{1}{8\pi^2 E_i} \int |M_{fi}|^2 \delta(m_i - E_1 - E_2) \delta^3(\vec{p}_1 + \vec{p}_2) \frac{d^3\vec{p}_1}{2E_1} \frac{d^3\vec{p}_2}{2E_2}$$

• Integrating over \vec{p}_2 using the δ -function:

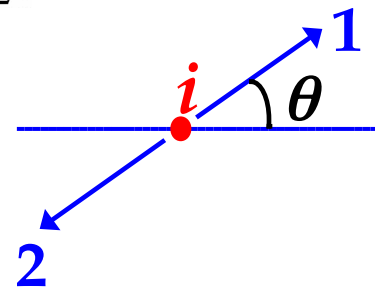
$$\Rightarrow \Gamma_{fi} = \frac{1}{8\pi^2 E_i} \int |M_{fi}|^2 \delta(m_i - E_1 - E_2) \frac{d^3\vec{p}_1}{4E_1 E_2}$$

now $E_2^2 = (m_2^2 + |\vec{p}_1|^2)$ since the δ -function imposes $\vec{p}_2 = -\vec{p}_1$

• Writing $d^3\vec{p}_1 = p_1^2 dp_1 \sin\theta d\theta d\phi = p_1^2 dp_1 d\Omega$

For convenience, here $|\vec{p}_1|$ is written as p_1

$$\Rightarrow \Gamma_{fi} = \frac{1}{32\pi^2 E_i} \int |M_{fi}|^2 \delta\left(m_i - \sqrt{m_1^2 + p_1^2} - \sqrt{m_2^2 + p_1^2}\right) \frac{p_1^2 dp_1 d\Omega}{E_1 E_2}$$



Two Body Decay – Cont.

- Which can be written in the form
$$\Gamma_{fi} = \frac{1}{32\pi^2 E_i} \int |M_{fi}|^2 g(p_1) \delta(f(p_1)) dp_1 d\Omega \quad (2)$$

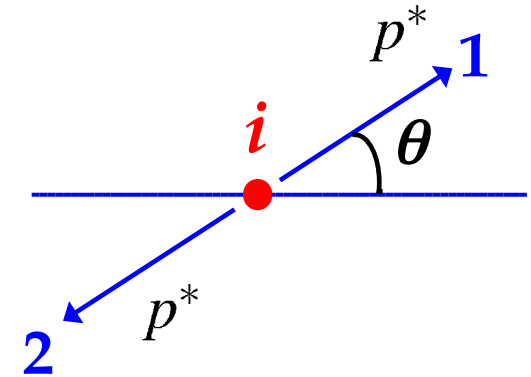
where $g(p_1) = p_1^2 / (E_1 E_2) = p_1^2 (m_1^2 + p_1^2)^{-1/2} (m_2^2 + p_1^2)^{-1/2}$

and $f(p_1) = m_i - (m_1^2 + p_1^2)^{1/2} - (m_2^2 + p_1^2)^{1/2}$

Note: $\delta(f(p_1))$ imposes energy conservation.

- $f(p_1) = 0$ determines the C.o.M momenta of the two decay products

i.e. $f(p_1) = 0$ for $p_1 = p^*$



- ★ Eq. (2) can be integrated using the property of δ – function derived earlier (eq. (1))

$$\int g(p_1) \delta(f(p_1)) dp_1 = \frac{1}{|df/dp_1|_{p^*}} \int g(p_1) \delta(p_1 - p^*) dp_1 = \frac{g(p^*)}{|df/dp_1|_{p^*}}$$

where p^* is the value for which $f(p^*) = 0$

- All that remains is to evaluate df/dp_1

$$\frac{df}{dp_1} = -\frac{p_1}{(m_1^2 + p_1^2)^{1/2}} - \frac{p_1}{(m_2^2 + p_1^2)^{1/2}} = -\frac{p_1}{E_1} - \frac{p_1}{E_2} = -p_1 \frac{E_1 + E_2}{E_1 E_2}$$

Two Body Decay – Cont.

giving:

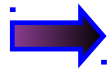
$$\Gamma_{fi} = \frac{1}{32\pi^2 E_i} \int |M_{fi}|^2 \left| \frac{E_1 E_2}{p_1(E_1 + E_2)} \frac{p_1^2}{E_1 E_2} \right|_{p_1=p^*} d\Omega$$

$$= \frac{1}{32\pi^2 E_i} \int |M_{fi}|^2 \left| \frac{p_1}{E_1 + E_2} \right|_{p_1=p^*} d\Omega$$

- But from $f(p_1) = 0$, i.e. energy conservation: $E_1 + E_2 = m_i$

$$\Gamma_{fi} = \frac{|\vec{p}^*|}{32\pi^2 E_i m_i} \int |M_{fi}|^2 d\Omega$$

In the particle's rest frame $E_i = m_i$




$$\frac{1}{\tau} = \Gamma = \frac{|\vec{p}^*|}{32\pi^2 m_i^2} \int |M_{fi}|^2 d\Omega \quad (3)$$

VALID FOR ALL TWO-BODY DECAYS !

- p^* can be obtained from $f(p_1) = 0$

$$(m_1^2 + p^{*2})^{1/2} + (m_2^2 + p^{*2})^{1/2} = m_i$$

 $p^* = \frac{1}{2m_i} \sqrt{[(m_i^2 - (m_1 + m_2)^2)] [m_i^2 - (m_1 - m_2)^2]}$ (Problem 3.2)

Cross section definition

$$\sigma = \frac{\text{no of interactions per unit time per target}}{\text{incident flux}}$$

Flux = number of incident particles/ unit area/unit time

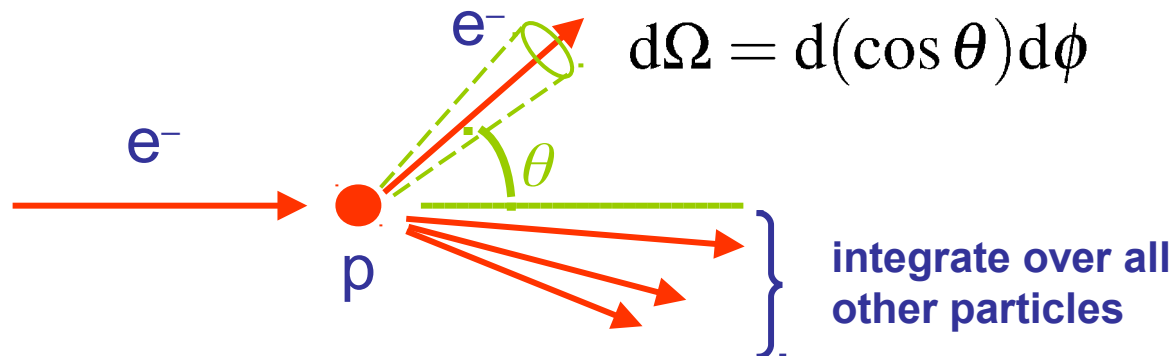
- The “cross section”, σ , can be thought of as the **effective** cross-sectional area of the target particles for the interaction to occur.
- In general this has nothing to do with the physical size of the target

Differential Cross section

$$\frac{d\sigma}{d\Omega} = \frac{\text{no of particles per sec/per target into } d\Omega}{\text{incident flux}}$$

or generally

$$\frac{d\sigma}{d\dots}$$

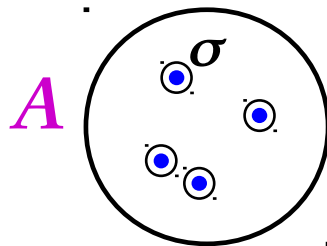
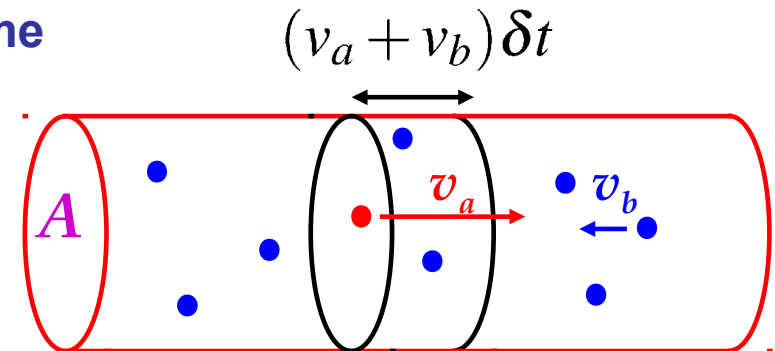


with
$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega$$

Cross Section Example

- Consider a single particle of type a with velocity, v_a , traversing a region of area A containing n_b particles of type b per unit volume

In time δt a particle of type a traverses region containing $n_b(v_a + v_b)A\delta t$ particles of type b



- ★ Interaction probability obtained from effective cross-sectional area occupied by the $n_b(v_a + v_b)A\delta t$ particles of type b

• Interaction Probability =
$$\frac{n_b(v_a + v_b)A\delta t\sigma}{A} = n_b v \delta t \sigma \quad [v = v_a + v_b]$$



Rate per particle of type $a = n_b v \sigma$

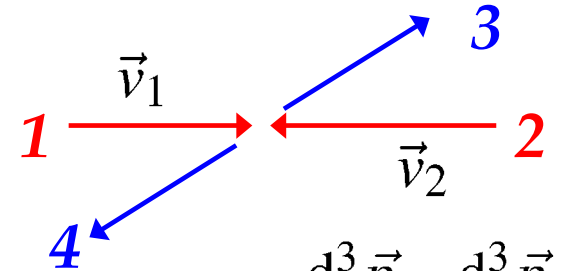
- Consider volume V , total reaction rate = $(n_b v \sigma) \cdot (n_a V) = (n_b V) (n_a v) \sigma = N_b \phi_a \sigma$

- As anticipated: Rate = Number of targets x Flux x cross section

Cross Section Calculations

- Consider scattering process

$$1 + 2 \rightarrow 3 + 4$$



- Start from Fermi's Golden Rule:

$$\Gamma_{fi} = (2\pi)^4 \int |T_{fi}|^2 \delta(E_1 + E_2 - E_3 - E_4) \delta^3(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4) \frac{d^3 \vec{p}_3}{(2\pi)^3} \frac{d^3 \vec{p}_4}{(2\pi)^3}$$

where T_{fi} is the transition matrix for a normalisation of 1/unit volume

- Now Rate/Volume = (flux of 1) \times (number density of 2) \times σ
 $= n_1(v_1 + v_2) \times n_2 \times \sigma$

- For 1 target particle per unit volume Rate = $(v_1 + v_2)\sigma$

$$\sigma = \frac{\Gamma_{fi}}{(v_1 + v_2)}$$

$$\sigma = \frac{(2\pi)^4}{v_1 + v_2} \int |T_{fi}|^2 \delta(E_1 + E_2 - E_3 - E_4) \delta^3(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4) \frac{d^3 \vec{p}_3}{(2\pi)^3} \frac{d^3 \vec{p}_4}{(2\pi)^3}$$

the parts are not Lorentz Invariant

- To obtain a Lorentz Invariant form use wave-functions normalised to $2E$ particles per unit volume

$$\psi' = (2E)^{1/2} \psi$$

- Again define **L.I. Matrix element** $M_{fi} = (2E_1 2E_2 2E_3 2E_4)^{1/2} T_{fi}$

$$\sigma = \frac{(2\pi)^{-2}}{2E_1 2E_2 (v_1 + v_2)} \int |M_{fi}|^2 \delta(E_1 + E_2 - E_3 - E_4) \delta^3(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4) \frac{d^3 \vec{p}_3}{2E_3} \frac{d^3 \vec{p}_4}{2E_4}$$

- The **integral** is now written in a Lorentz invariant form
- The quantity $F = 2E_1 2E_2 (v_1 + v_1)$ can be written in terms of a four-vector scalar product and is therefore also Lorentz Invariant (the Lorentz Inv. Flux)

$$F = 4 \left[(p_1^\mu p_{2\mu})^2 - m_1^2 m_2^2 \right]^{1/2} \quad \text{(see appendix I)}$$

- Consequently cross section is a **Lorentz Invariant** quantity

Two special cases of Lorentz Invariant Flux:

- **Centre-of-Mass Frame**

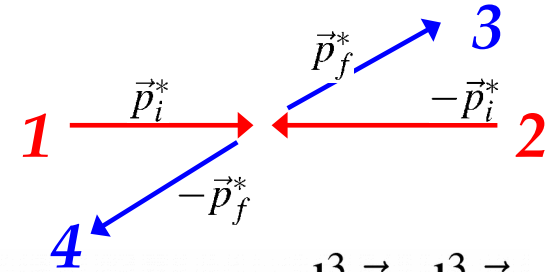
$$\begin{aligned} F &= 4E_1 E_2 (v_1 + v_2) \\ &= 4E_1 E_2 (|\vec{p}^*|/E_1 + |\vec{p}^*|/E_2) \\ &= 4|\vec{p}^*| (E_1 + E_2) \\ &= 4|\vec{p}^*| \sqrt{s} \end{aligned}$$

- **Target (particle 2) at rest**

$$\begin{aligned} F &= 4E_1 E_2 (v_1 + v_2) \\ &= 4E_1 m_2 v_1 \\ &= 4E_1 m_2 (|\vec{p}_1|/E_1) \\ &= 4m_2 |\vec{p}_1| \end{aligned}$$

2 2 Body Scattering in C.o.M. Frame

- We will now apply above Lorentz Invariant formula for the interaction cross section to the most common cases used in the course. First consider 2 2 scattering in C.o.M. frame



- Start from

$$\sigma = \frac{(2\pi)^{-2}}{2E_1 2E_2 (v_1 + v_2)} \int |M_{fi}|^2 \delta(E_1 + E_2 - E_3 - E_4) \delta^3(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4) \frac{d^3 \vec{p}_3}{2E_3} \frac{d^3 \vec{p}_4}{2E_4}$$

- Here $\vec{p}_1 + \vec{p}_2 = 0$ and $E_1 + E_2 = \sqrt{s}$

$$\Rightarrow \sigma = \frac{(2\pi)^{-2}}{4|\vec{p}_i^*| \sqrt{s}} \int |M_{fi}|^2 \delta(\sqrt{s} - E_3 - E_4) \delta^3(\vec{p}_3 + \vec{p}_4) \frac{d^3 \vec{p}_3}{2E_3} \frac{d^3 \vec{p}_4}{2E_4}$$

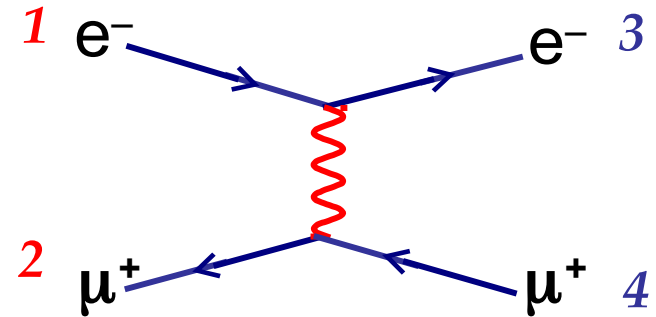
- ★ The integral is exactly the same integral that appeared in the particle decay calculation but with m_a replaced by \sqrt{s}

$$\Rightarrow \sigma = \frac{(2\pi)^{-2}}{4|\vec{p}_i^*| \sqrt{s}} \frac{|\vec{p}_f^*|}{4\sqrt{s}} \int |M_{fi}|^2 d\Omega^*$$

$$\sigma = \frac{1}{64\pi^2 s} \frac{|\vec{p}_f^*|}{|\vec{p}_i^*|} \int |M_{fi}|^2 d\Omega^*$$

- In the case of elastic scattering $|\vec{p}_i^*| = |\vec{p}_f^*|$

$$\sigma_{\text{elastic}} = \frac{1}{64\pi^2 s} \int |M_{fi}|^2 d\Omega^*$$



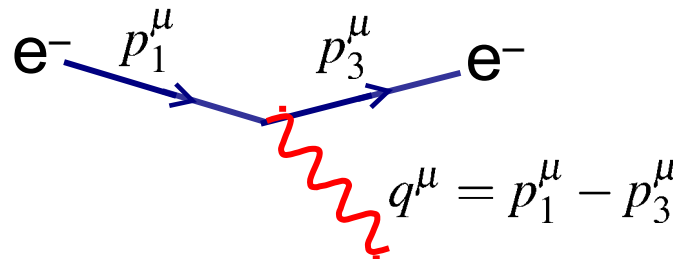
- For calculating the total cross-section (which is Lorentz Invariant) the result on the previous page (eq. (4)) is sufficient. However, it is not so useful for calculating the differential cross section in a rest frame other than the C.o.M:

$$d\sigma = \frac{1}{64\pi^2 s} \frac{|\vec{p}_f^*|}{|\vec{p}_i^*|} |M_{fi}|^2 d\Omega^*$$

because the angles in $d\Omega^* = d(\cos \theta^*) d\phi^*$ refer to the C.o.M frame

- For the last calculation in this section, we need to find a L.I. expression for $d\sigma$
- ★ Start by expressing $d\Omega^*$ in terms of Mandelstam t i.e. the square of the four-momentum transfer

$$t = q^2 = (p_1 - p_3)^2$$



Product of four-vectors therefore L.I.

- Want to express $d\Omega^*$ in terms of **Lorentz Invariant** dt
 where $t \equiv (p_1 - p_3)^2 = p_1^2 + p_3^2 - 2p_1 \cdot p_3 = m_1^2 + m_3^2 - 2p_1 \cdot p_3$

- ♦ In C.o.M. frame:

$$p_1^{*\mu} = (E_1^*, 0, 0, |\vec{p}_1^*|)$$

$$p_3^{*\mu} = (E_3^*, |\vec{p}_3^*| \sin \theta^*, 0, |\vec{p}_3^*| \cos \theta^*)$$

$$p_1^\mu p_{3\mu} = E_1^* E_3^* - |\vec{p}_1^*| |\vec{p}_3^*| \cos \theta^*$$

$$t = m_1^2 + m_3^2 - E_1^* E_3^* + 2|\vec{p}_1^*| |\vec{p}_3^*| \cos \theta^*$$

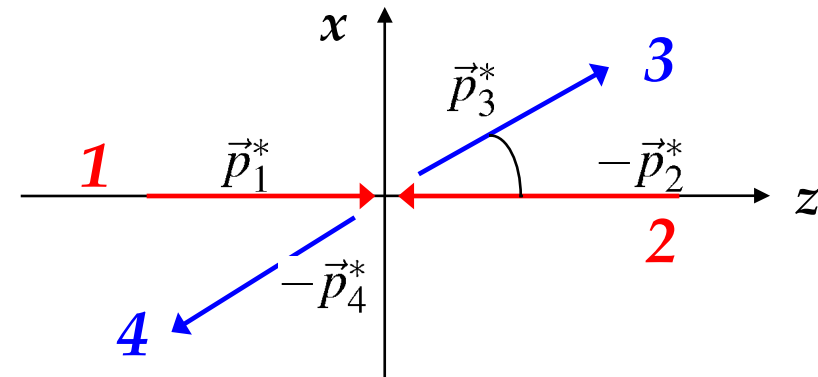
giving $dt = 2|\vec{p}_1^*| |\vec{p}_3^*| d(\cos \theta^*)$

therefore $d\Omega^* = d(\cos \theta^*) d\phi^* = \frac{dt d\phi^*}{2|\vec{p}_1^*| |\vec{p}_3^*|}$

hence $d\sigma = \frac{1}{64\pi^2 s} \frac{|\vec{p}_3^*|}{|\vec{p}_1^*|} |M_{fi}|^2 d\Omega^* = \frac{1}{2 \cdot 64\pi^2 s |\vec{p}_1^*|^2} |M_{fi}|^2 d\phi^* dt$

- Finally, integrating over $d\phi^*$ (assuming no ϕ^* dependence of $|M_{fi}|^2$) gives:

$$\frac{d\sigma}{dt} = \frac{1}{64\pi s |\vec{p}_i^*|^2} |M_{fi}|^2$$



Lorentz Invariant differential cross section

- All quantities in the expression for $d\sigma/dt$ are Lorentz Invariant and therefore, it applies to **any rest frame**. It should be noted that $|\vec{p}_i^*|^2$ is a constant, fixed by energy/momentum conservation

$$|\vec{p}_i^*|^2 = \frac{1}{4s} [s - (m_1 + m_2)^2][s - (m_1 - m_2)^2]$$

- As an example of how to use the invariant expression $d\sigma/dt$ we will consider elastic scattering in the laboratory frame in the limit where we can neglect the mass of the incoming particle $E_1 \gg m_1$



m_2

e.g. electron or neutrino scattering

In this limit

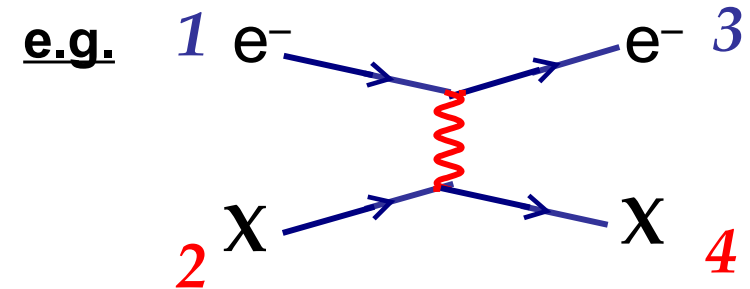
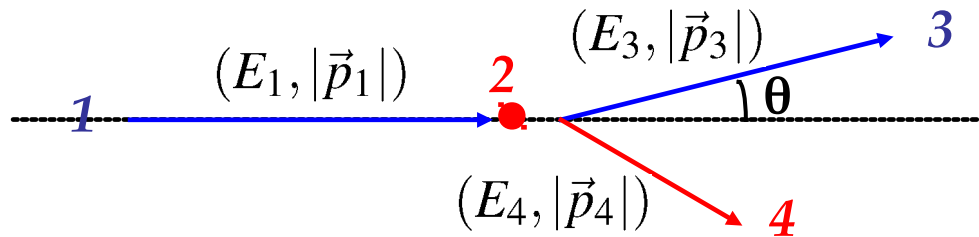
$$|\vec{p}_i^*|^2 = \frac{(s - m_2)^2}{4s}$$

$$\frac{d\sigma}{dt} = \frac{1}{16\pi(s - m_2^2)^2} |M_{fi}|^2$$

$$(m_1 = 0)$$

2.2 Body Scattering in Lab. Frame

- The other commonly occurring case is scattering from a fixed target in the Laboratory Frame (e.g. electron-proton scattering)
- First take the case of elastic scattering at high energy where the mass of the incoming particles can be neglected: $m_1 = m_3 = 0$, $m_2 = m_4 = M$



- Wish to express the cross section in terms of scattering angle of the e^-

$$\text{therefore} \quad \frac{d\sigma}{d\Omega} = \frac{d\sigma}{dt} \frac{dt}{d\Omega} = \frac{1}{2\pi} \frac{dt}{d(\cos \theta)} \frac{d\sigma}{dt}$$

Integrating over $d\phi$

- The rest is some rather tedious algebra... start from four-momenta

$$p_1 = (E_1, 0, 0, E_1), \quad p_2 = (M, 0, 0, 0), \quad p_3 = (E_3, E_3 \sin \theta, 0, E_3 \cos \theta), \quad p_4 = (E_4, \vec{p}_4)$$

so here $t = (p_1 - p_3)^2 = -2p_1 \cdot p_3 = -2E_1 E_3 (1 - \cos \theta)$

But from (E,p) conservation $p_1 + p_2 = p_3 + p_4$

and, therefore, can also express t in terms of particles 2 and 4

$$\begin{aligned}
 t &= (p_2 - p_4)^2 = 2M^2 - 2p_2 \cdot p_4 = 2M^2 - 2ME_4 \\
 &= 2M^2 - 2M(E_1 + M - E_3) = -2M(E_1 - E_3)
 \end{aligned}$$

Note E_1 is a constant (the energy of the incoming particle) so

$$\frac{dt}{d(\cos \theta)} = 2M \frac{dE_3}{d(\cos \theta)}$$

• Equating the two expressions for t gives

$$E_3 = \frac{E_1 M}{M + E_1 - E_1 \cos \theta}$$

so

$$\frac{dE_3}{d(\cos \theta)} = \frac{E_1 M}{(M + E_1 - E_1 \cos \theta)^2} = E_1^2 M \left(\frac{E_3}{E_1 M} \right)^2 = \frac{E_3^2}{M}$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{2\pi} \frac{dt}{d(\cos \theta)} \frac{d\sigma}{dt} = \frac{1}{2\pi} 2M \frac{E_3^2}{M} \frac{d\sigma}{dt} = \frac{E_3^2}{\pi} \frac{d\sigma}{dt} = \frac{E_3^2}{\pi} \frac{1}{16\pi(s - M^2)^2} |M_{fi}|^2$$

using gives

$$\begin{aligned}
 s &= (p_1 + p_2)^2 = M^2 + 2p_1 \cdot p_2 = M^2 + 2ME_1 \\
 (s - M^2) &= 2ME_1
 \end{aligned}$$

Particle 1 massless
 $\rightarrow (p_1^2 = 0)$

$$\rightarrow \frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \left(\frac{E_3}{ME_1} \right)^2 |M_{fi}|^2$$

In limit $m_1 \rightarrow 0$

In this equation, E_3 is a function of θ :

$$E_3 = \frac{E_1 M}{M + E_1 - E_1 \cos \theta}$$

giving

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \left(\frac{1}{M + E_1 - E_1 \cos \theta} \right)^2 |M_{fi}|^2 \quad (m_1 = 0)$$

General form for 2-2 Body Scattering in Lab. Frame

★ The calculation of the differential cross section for the case where m_1 can not be neglected is longer and contains no more “physics” (see appendix II). It gives:

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \cdot \frac{1}{p_1 m_1} \cdot \frac{|\vec{p}_3|^2}{|\vec{p}_3|(E_1 + m_2) - E_3 |\vec{p}_1| \cos \theta} \cdot |M_{fi}|^2$$

Again there is only one independent variable, θ , which can be seen from conservation of energy

$$E_1 + m_2 = \sqrt{|\vec{p}_3|^2 + m_3^2} + \sqrt{|\vec{p}_1|^2 + |\vec{p}_3|^2 - 2|\vec{p}_1||\vec{p}_3|\cos\theta + m_4^2}$$

i.e. $|\vec{p}_3|$ is a function of θ

$$\vec{p}_4 = \vec{p}_1 - \vec{p}_3$$

Summary

- ★ Used a Lorentz invariant formulation of Fermi's Golden Rule to derive decay rates and cross-sections in terms of the **Lorentz Invariant Matrix Element** (wave-functions normalised to $2E/\text{Volume}$)

Main Results:

- ★ Particle decay:

$$\Gamma = \frac{|\vec{p}^*|}{32\pi^2 m_i^2} \int |M_{fi}|^2 d\Omega$$

Where p^* is a function of particle masses

$$p^* = \frac{1}{2m_i} \sqrt{[(m_i^2 - (m_1 + m_2)^2)] [m_i^2 - (m_1 - m_2)^2]}$$

- ★ Scattering cross section in C.o.M. frame:

$$\sigma = \frac{1}{64\pi^2 s} \frac{|\vec{p}_f^*|}{|\vec{p}_i^*|} \int |M_{fi}|^2 d\Omega^*$$

- ★ Invariant differential cross section (valid in all frames):

$$\frac{d\sigma}{dt} = \frac{1}{64\pi s |\vec{p}_i^*|^2} |M_{fi}|^2$$

$$|\vec{p}_i^*|^2 = \frac{1}{4s} [s - (m_1 + m_2)^2] [s - (m_1 - m_2)^2]$$

Summary cont.

★ Differential cross section in the lab. frame ($m_1=0$)

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \left(\frac{E_3}{ME_1} \right)^2 |M_{fi}|^2$$



$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \left(\frac{1}{M + E_1 - E_1 \cos \theta} \right)^2 |M_{fi}|^2$$

★ Differential cross section in the lab. frame ($m_1 \neq 0$)

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \cdot \frac{1}{|\vec{p}_1| m_1} \cdot \frac{|\vec{p}_3|^2}{|\vec{p}_3| (E_1 + m_2) - E_3 |\vec{p}_1| \cos \theta} \cdot |M_{fi}|^2$$

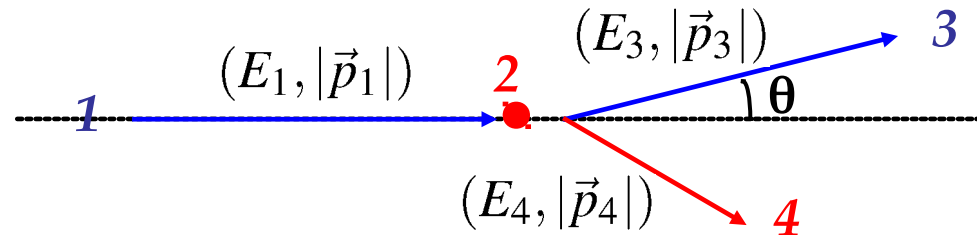
with $E_1 + m_2 = \sqrt{|\vec{p}_3|^2 + m_3^2} + \sqrt{|\vec{p}_1|^2 + |\vec{p}_3|^2 - 2|\vec{p}_1||\vec{p}_3| \cos \theta + m_4^2}$

Summary of the summary:

- ★ Have now dealt with **kinematics** of particle decays and cross sections
- ★ The **fundamental particle physics** is in the matrix element
- ★ The above equations are the basis for all calculations that follow

Appendix II : general 2-2 Body Scattering in lab frame

NON-EXAMINABLE



$$p_1 = (E_1, 0, 0, |\vec{p}_1|), \quad p_2 = (M, 0, 0, 0), \quad p_3 = (E_3, E_3 \sin \theta, 0, E_3 \cos \theta), \quad p_4 = (E_4, \vec{p}_4)$$

again

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{dt} \frac{dt}{d\Omega} = \frac{1}{2\pi} \frac{dt}{d(\cos \theta)} \frac{d\sigma}{dt}$$

But now the invariant quantity t :

$$\begin{aligned} t &= (p_2 - p_4)^2 = m_2^2 + m_4^2 - 2p_2 \cdot p_4 = m_2^2 + m_4^2 - 2m_2 E_4 \\ &= m_2^2 + m_4^2 - 2m_2 (E_1 + m_2 - E_3) \end{aligned}$$

$$\rightarrow \frac{dt}{d(\cos \theta)} = 2m_2 \frac{dE_3}{d(\cos \theta)}$$

Which gives
$$\frac{d\sigma}{d\Omega} = \frac{m_2}{\pi} \frac{dE_3}{d(\cos\theta)} \frac{d\sigma}{dt}$$

To determine $dE_3/d(\cos\theta)$, first differentiate $E_3^2 - |\vec{p}_3|^2 = m_3^2$


$$2E_3 \frac{dE_3}{d(\cos\theta)} = 2|\vec{p}_3| \frac{d|\vec{p}_3|}{d(\cos\theta)} \quad (\text{AII.1})$$

Then equate $t = (p_1 - p_3)^2 = (p_4 - p_2)^2$ to give

$$m_1^2 + m_3^2 - 2(E_1 E_3 - |\vec{p}_1| |\vec{p}_3| \cos\theta) = m_4^2 + m_2^2 - 2m_2(E_1 + m_2 - E_3)$$

Differentiate wrt. $\cos\theta$

$$(E_1 + m_2) \frac{dE_3}{d\cos\theta} - |\vec{p}_1| \cos\theta \frac{d|\vec{p}_3|}{d\cos\theta} = |\vec{p}_1| |\vec{p}_3|$$

Using (1) 
$$\frac{dE_3}{d(\cos\theta)} = \frac{|\vec{p}_1| |\vec{p}_3|^2}{|\vec{p}_3| (E_1 + m_2) - E_3 |\vec{p}_1| \cos\theta} \quad (\text{AII.2})$$

$$\frac{d\sigma}{d\Omega} = \frac{m_2}{\pi} \frac{dE_3}{d(\cos\theta)} \frac{d\sigma}{dt} = \frac{m_2}{\pi} \frac{dE_3}{d(\cos\theta)} \frac{1}{64\pi s |\vec{p}_i^*|^2} |M_{fi}|^2$$

It is easy to show $|\vec{p}_i^*| \sqrt{s} = m_2 |\vec{p}_1|$

$$\frac{d\sigma}{d\Omega} = \frac{dE_3}{d(\cos \theta)} \frac{m_2}{64\pi^2 m_2^2 |\vec{p}_1|^2} |M_{fi}|^2$$

and using (AII.2) obtain

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \cdot \frac{1}{p_1 m_1} \cdot \frac{|\vec{p}_3|^2}{|\vec{p}_3| (E_1 + m_2) - E_3 |\vec{p}_1| \cos \theta} \cdot |M_{fi}|^2$$

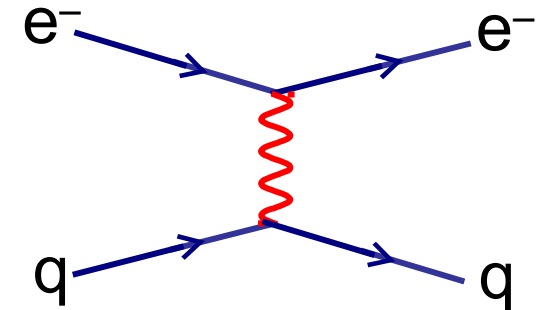
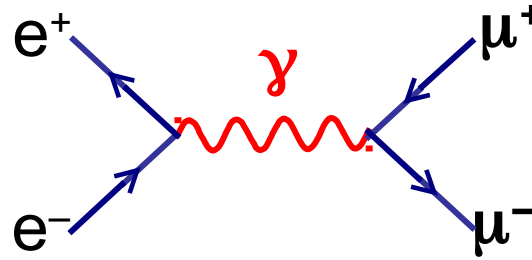
Next

- ★ Aiming towards a proper calculation of decay and scattering processes

Will concentrate on:

- $e^+e^- \rightarrow \mu^+\mu^-$
- $e^-q \rightarrow e^-q$

($e^-q \rightarrow e^-q$ to probe proton structure)



- ★ Need relativistic calculations of particle decay rates and cross sections:

$$\sigma = \frac{|M_{fi}|^2}{\text{flux}} \times (\text{phase space})$$

- ★ Need relativistic treatment of spin-half particles:

Dirac Equation

- ★ Need relativistic calculation of interaction Matrix Element:

Interaction by particle exchange and Feynman rules

- + and a few mathematical tricks along, e.g. the Dirac Delta Function