MMP 2
Solutions 2

FS 19
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Exercise 1

Identities:

$$
\begin{aligned}
|z| &= \sqrt{z\bar{z}} \\
|\sqrt{z}| &= \sqrt{|z|} \\
|w| \leq |z| &\Rightarrow |w|^2 \leq |z|^2 \\
|z| &= \sqrt{|z|}^2 \\
Re(z) &= \frac{z+\bar{z}}{2} \\
Im(z) &= \frac{z-\bar{z}}{2} \\
|z_1 z_2| &= |z_1||z_2| \\
z &= x + yi \\
|z| &= |\bar{z}|
\end{aligned}
$$

a) i)

$$
\begin{aligned}
|z_1 + z_2| &\leq |z_1| + |z_2| && |^{\wedge 2} \\
(z_1+z_2)(\bar{z}_1+\bar{z}_2) &\leq |z_1|^2 + |z_2|^2 + 2|z_1||z_2| \\
|z_1|^2 + |z_2|^2 + \bar{z}_1 z_2 + z_1 \bar{z}_2 &\leq |z_1|^2 + |z_2|^2 + 2\sqrt{z_1\bar{z}_1 z_2 \bar{z}_2} \\
\bar{z}_1 z_2 + z_1 \bar{z}_2 &\leq 2\sqrt{z_1\bar{z}_1 z_2 \bar{z}_2} && \mid z := z_1\bar{z}_2 \\
z + \bar{z} &\leq 2\sqrt{z\bar{z}} && :2 \\
Re(z) &\leq |z| = \sqrt{Re(z)^2 + Im(z)^2} && \square
\end{aligned}
$$

ii) use (i):

$$
\begin{aligned}
|\underbrace{z_1 + z_2} + (-z_2)| &\leq |z_1 + z_2| + |z_2| && |-|z_2| \\
|z_1| - |z_2| &\leq |z_1 + z_2| && \square
\end{aligned}
$$

iii)

$$
\begin{aligned}
||z_1| - |z_2|| &\leq |z_1 - z_2| && |^\wedge 2 \\
|z_1|^2 + |z_2|^2 - 2|z_1||z_2| &\leq (z_1 - z_2)(\bar{z}_1 - \bar{z}_2) \\
&= z_1\bar{z}_1 + \bar{z}_2 z_2 - \bar{z}_1 z_2 - z_1\bar{z}_2 && z := z_1\bar{z}_2 \\
-2|z| &\leq -z - \bar{z} \\
\Rightarrow Re(z) &\leq |z| && \square
\end{aligned}
$$

b)

$$
\begin{aligned}
|z-1| &< \sqrt{|z^2 - 1|} < |z+1| && |^\wedge 2 \\
|z-1|^2 &< |z^2 - 1| = |z-1||z+1| < |z+1|^2
\end{aligned}
$$

$$
\begin{aligned}
|z-1|^2 &< |z-1||z+1| : && \Rightarrow z \neq \pm 1 \\
|z-1| &< |z+1| && \Rightarrow z \neq 0 \\
\sqrt{z\bar{z} - z - \bar{z} + 1} &< \sqrt{z\bar{z} + z + \bar{z} + 1} && |^\wedge 2 \\
-2Re(z) &< 2Re(z) \\
\Rightarrow Re(z) &> 0
\end{aligned}
$$

$|z-1||z+1| < |z+1|^2 :$ is valid with the conditions set in the first part

Conditions:

$$
\begin{aligned}
Re(z) &> 0 \\
z &\neq \pm 1
\end{aligned}
$$

Exercise 2

a)

$$\sinh(iz) = \sum_{s=0}^{\infty} \frac{(iz)^{2s+1}}{(2s+1)!} = i \sum_{s=0}^{\infty} i^{2s} \frac{z^{2s+1}}{(2s+1)!} = i \sum_{s=0}^{\infty} (-1)^s \frac{z^{2s+1}}{(2s+1)!} = i \sin(z)$$

$$\cosh(iz) = \sum_{s=0}^{\infty} \frac{(iz)^{2s}}{(2s)!} = \sum_{s=0}^{\infty} (-1)^s \frac{z^{2s}}{(2s)!} = \cos(z)$$

$$\sin(iz) = \sum_{s=0}^{\infty} (-1)^s \frac{(iz)^{2s+1}}{(2s+1)!} = i \sum_{s=0}^{\infty} \underbrace{(-1)^{2s}}_{=1} \frac{z^{2s+1}}{(2s+1)!} = i \sum_{s=0}^{\infty} \frac{z^{2s+1}}{(2s+1)!} = i \sinh(z)$$

$$\cos(iz) = \sum_{s=0}^{\infty} (-1)^s \frac{(iz)^{2s}}{(2s)!} = \sum_{s=0}^{\infty} (-1)^{2s} \frac{z^{2s}}{(2s)!} = \sum_{s=0}^{\infty} \frac{z^{2s}}{(2s)!} = \cosh(z)$$

b)

$$\frac{e^z + e^{-z}}{2} = \frac{1}{2} \left(\sum_{s=0}^{\infty} \frac{z^s}{s!} + \frac{(-z)^s}{s!} \right) = \frac{1}{2} \sum_{s=0}^{\infty} \frac{z^s}{s!} (1 + (-1)^s) = \frac{1}{2} \sum_{n=1, even}^{\infty} 2 \frac{z^n}{n!} =$$

$$= \sum_{s=0}^{\infty} \frac{z^{2s}}{(2s)!} = \cosh(z)$$

$$\frac{e^z - e^{-z}}{2} = \frac{1}{2} \sum_{s=0}^{\infty} \frac{z^s}{s!} (1 - (-1)^s) = \frac{1}{2} \sum_{n=1, odd}^{\infty} 2 \frac{z^n}{n!} = \sum_{s=0}^{\infty} \frac{z^{2s+1}}{(2s+1)!} = \sinh(z)$$

Exercise 3

a)

$$
\begin{aligned}
&\sin(x+yi) = \frac{e^{i(x+yi)} - e^{-i(x+yi)}}{2i} \\
&\sin(x)\cosh(y) + i\cos(x)\sinh(y) = \\
&= \left(\frac{e^{ix} - e^{-ix}}{2i}\right)\left(\frac{e^{y} + e^{-y}}{2}\right) + i\left(\frac{e^{ix} + e^{-ix}}{2}\right)\left(\frac{e^{y} - e^{-y}}{2}\right) = \\
&= \frac{1}{4}\left[\frac{1}{i}\left(e^{ix+y} + e^{ix-y} - e^{-ix+y} - e^{-ix-y}\right) + i\left(e^{ix+y} - e^{ix-y} + e^{-ix+y} - e^{-ix-y}\right)\right] = \\
&= \frac{1}{4i}\left[e^{ix+y} + e^{ix-y} - e^{-ix+y} - e^{-ix-y} - e^{ix+y} + e^{ix-y} - e^{-ix+y} + e^{-ix-y}\right] = \\
&= \frac{1}{4i}\left[2e^{ix-y} - 2e^{-ix+y}\right] = \\
&= \frac{1}{2i}\left[e^{ix-y} - e^{-ix+y}\right] \\
&= \frac{1}{2i}\left[e^{i(x+yi)} - e^{-i(x+yi)}\right]
\end{aligned}
$$

$\square$

$\cos(x+yi) = \cos(x)\cosh(y) - i\sin(x)\sinh(y)$ in the same way!

b)

$$
\begin{aligned}
|\sin(x+yi)|^2 &= |\sin x\cosh y + i\cos x\sinh y|^2 = \sin^2 x\cosh^2 y + \cos^2 x\sinh^2 y \\
&= \sin^2 x(1+\sinh^2 y) + (1-\sin^2 x)\sinh^2 y \\
&= \sin^2 x + \sin^2 x\sinh^2 y + \sinh^2 y - \sin^2 x\sinh^2 y \\
&= \sin^2 x + \sinh^2 y \\
|\cos(x+yi)|^2 &= |\cos x\cosh y - i\sin x\sinh y|^2 = \cos^2 x\cosh^2 y + \sin^2 x\sinh^2 y \\
&= \cos^2 x(1+\sinh^2 y) + (1-\cos^2 x)\sinh^2 y \\
&= \cos^2 x + \sinh^2 y
\end{aligned}
$$

Exercise 4

$$
\begin{aligned}
|\sin z| &\overset{3b)}{=} \sqrt{\sin^2 x + \underbrace{\sinh^2 y}_{\geq 0}} \geq \sqrt{\sin^2 x} = |\sin x| \\
|\cos z| &= \sqrt{\cos^2 x + \sinh^2 y} \geq |\cos x|
\end{aligned}
$$

$\square$