MMP 2
Solutions 5

FS 19
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Exercise 1

$$
\begin{aligned}
z &= e^{i\varphi} && \text{(unit circle)}\\
dz &= ie^{i\varphi}d\varphi\\
\int_\gamma Re(z^n)dz &= \int_0^{2\pi} Re\left(e^{in\varphi}\right) ie^{i\varphi}d\varphi = i\int_0^{2\pi}\frac{e^{in\varphi}+e^{-in\varphi}}{2}e^{i\varphi}d\varphi\\
&= \frac{i}{2}\left[\int_0^{2\pi} e^{i(n+1)\varphi}d\varphi + \int_0^{2\pi} e^{-i(n-1)\varphi}d\varphi\right]\\
&= \frac{i}{2}\begin{cases} n=-1: & \int_0^{2\pi} d\varphi + \underbrace{\int_0^{2\pi} e^{2i\phi}d\varphi}_{=0} = 2\pi\\ n=+1: & \underbrace{\int_0^{2\pi} e^{2i\varphi}d\varphi}_{=0} + \int_0^{2\pi} d\varphi = 2\pi\\ n\neq\pm1: & \underbrace{\frac{1}{i(n+1)}e^{i(n+1)}\Big|_0^{2\pi}}_{=0} + \underbrace{\frac{1}{-i(n-1)}e^{-i(n-1)}\Big|_0^{2\pi}}_{=0} = 0\end{cases}\\
&= \begin{cases} n=-1,1: & i\pi\\ n\neq-1,1: & 0\end{cases}
\end{aligned}
$$

Exercise 2

$$
\begin{aligned}
f(z) &= (z+i)^{-\frac{1}{5}}\\
\Rightarrow f'(z) &= -\frac{1}{5}(z+i)^{-(\frac{1}{5}+1)}\\
f''(z) &= +\frac{1}{5}\left(\frac{1}{5}+1\right)(z+i)^{-(\frac{1}{5}+2)}\\
\Rightarrow f^{(k)}(z) &= (-1)^k(z+i)^{-(\frac{1}{5}+k)}\cdot\prod_{n=0}^{k-1}\left(\frac{1}{5}+n\right)
\end{aligned}
$$

Taylor series around $z = a$ $(a \neq i)$:

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2}(z-a)^2 + ...$$

Since $\sqrt[5]{z}$ has 5 solutions and every summand could use a different root (then our series would not be well-defined), we fix to the zero-order solution, i.e. $\sqrt[5]{z} \equiv \sqrt[5]{r}e^{i\frac{\varphi}{5}}$

Hudamard's formula for the radius of convergence:

$$\begin{aligned}
\rho &= \lim_{k\to\infty}\left|\frac{c_k}{c_{k+1}}\right| = \lim_{k\to\infty}\left|\frac{\frac{1}{k!}f^{(k)}(a)}{\frac{1}{(k+1)!}f^{(k+1)}(a)}\right| \\
f(a) &= \sum_{k=0}^{\infty} c_k(z-a)^k \\
\rho &= \lim_{k\to\infty}\left|\frac{\frac{1}{k!}(-1)^k(a+i)^{-(\frac{1}{5}+k)}\prod_{n=0}^{k-1}\left(\frac{1}{5}+n\right)}{\frac{1}{(k+1)!}(-1)^{k+1}(a+i)^{-(\frac{1}{5}+k+1)}\prod_{n=0}^{k}\left(\frac{1}{5}+n\right)}\right| \\
&= \lim_{k\to\infty}\underbrace{\left|(-1)^{-1}\right|}_{=1}\left|(k+1)(a+i)^{-\frac{1}{5}-k+\frac{1}{5}+k+1}\frac{1}{k+\frac{1}{5}}\right| \\
&= \underbrace{\lim_{k\to\infty}\left|\frac{k+1}{k+\frac{1}{5}}\right|}_{\to 1}|a+i| = |a+i|
\end{aligned}$$

Exercise 3

Laurent series (around z = c):

$$\begin{aligned}
f(z) &= \sum_{n=-\infty}^{\infty} a_n(z-c)^n \\
f(z) &= \frac{1}{z^4-1}
\end{aligned}$$

around $z = 1$

$\rightarrow$ Expand $f(z)$ into its rots:

$$
\begin{aligned}
& z^4 - 1 = 0 \Rightarrow z^4 = 1 \Rightarrow z = 1, -1, i, -i \\
\Rightarrow\ & z^4 - 1 = (z+1)(z-1)(z+i)(z-i) \\
\Rightarrow\ & \frac{1}{z^4-1} = \frac{A}{z+1} + \frac{B}{z-1} + \frac{C}{z+i} + \frac{D}{z-i} && \mid \cdot (z^4 - 1) \\
& \Rightarrow 1 = z^3(A+B+C+D) + z^2(-A+B-iC+iD) + \\
& \quad + z(A+B-C-D) + (-A+B+iC-iD) \\
& \Rightarrow \begin{cases} A+B+C=0 \\ -A+B-iC+iD=0 \\ A+B-C-D=0 \\ -A+B+iC-iD=1 \end{cases} \rightarrow \begin{cases} A=-B \\ C=-D \\ B=iD \\ 4B=1 \end{cases} \Rightarrow \begin{cases} A=-\frac{1}{4} \\ B=\frac{1}{4} \\ C=-\frac{i}{4} \\ D=\frac{i}{4} \end{cases} \\
\Rightarrow\ & \frac{1}{z^4-1} = -\frac{1}{4}\frac{1}{z+1} + \frac{1}{4}\frac{1}{z-1} - \frac{i}{4}\frac{1}{z+i} + \frac{i}{4}\frac{1}{z-i}
\end{aligned}
$$

$\frac{1}{4}\frac{1}{z-1}$ is already part of a Laurent series ($n = -1, a_{-1} = \frac{1}{4}$)

$\Rightarrow$ Try to expand the remaining 3 summands in Laurent series around 1

We know that for a geometric series:

$$\frac{1}{1-r} = \sum_{k=0}^{\infty} r^k,\ r \in \mathbb{C},\ |r| < 1$$

$\Rightarrow$ Try to bring $\frac{1}{z+1}$ to that form

$$\rightarrow \frac{1}{z+1} = \frac{1}{2(1-\frac{1-z}{2})} = \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{1-z}{2}\right)^k$$

for $\left|\frac{1-z}{2}\right| < 1 \Rightarrow |1-z| < 2$

on the other hand:

$$\frac{1}{z-i} = \frac{1}{(z-1)+(1-i)} = \frac{1}{1-i}\frac{1}{1+\frac{z-1}{1-i}} = \frac{1}{1-i}\frac{1}{1-\frac{z-1}{i-1}} = \frac{1}{1-i}\sum_{k=0}^{\infty}\left(\frac{z-1}{i-1}\right)^k$$

for $\left|\frac{z-1}{1-i}\right| < 1 \Rightarrow |z-1| < \sqrt{2}$

$$\frac{1}{z+i} = \frac{1}{(z-1)+(1+i)} = \frac{1}{1+i}\frac{1}{1+\frac{z-1}{1+i}} = \frac{1}{1+i}\frac{1}{1-\frac{1-z}{i+1}} = \frac{1}{1+i}\sum_{k=0}^{\infty}(-1)^k\left(\frac{z-1}{i+1}\right)^k$$

for $\left|\frac{z-1}{1+i}\right| < 1 \Rightarrow |z-1| < \sqrt{2}$

$$\Rightarrow \frac{1}{z^4-1} = \frac{1}{4}(z-1)^{-1} + \sum_{k=0}^{\infty}\left[-\frac{1}{4}\frac{1}{2}\left(\frac{1-z}{2}\right)^k + \frac{i}{4}\frac{1}{1-i}\left(\frac{z-1}{i-1}\right)^k - \frac{i}{4}\frac{1}{1+i}(-1)^k\left(\frac{z-1}{i+1}\right)^k\right]$$

$$= \frac{1}{4}(z-1)^{-1} + \frac{1}{4}\sum_{k=0}^{\infty}(z-1)^k\left[\frac{-(-1)^k}{2^{k+1}} - \frac{i}{(i-1)^{k+1}} - \frac{i}{(i+1)^{k+1}}(-1)^k\right]$$

$$= \frac{1}{4}(z-1)^{-1} + \frac{1}{4}\sum_{k=0}^{\infty}(z-1)^k\left[-\frac{i(i+1)^k + i(i-1)^k(-1)^k}{[\underbrace{(i-1)(i+1)}_{=-2}]^{k+1}}\right]$$

$$= \underbrace{\frac{1}{4}}_{a_{-1}}(z-1)^{-1} + \sum_{k=0}^{\infty}\underbrace{\frac{1}{4(-2)^{k+1}}\left[1 - i(i+1)^{k+1} + (-1)^k i(i-1)^{k+1}\right]}_{a_k}(z-1)^k \qquad \square$$

$\Rightarrow$ Laurent series around $z = 1$

Convergence radius: $|z-1| < \sqrt{2}$

Exercise 4

$$I = \lim_{\varepsilon\to 0}\left(\int_0^{1-\varepsilon} + \int_{1+\varepsilon}^{\infty}\right)\frac{e^{iax}+e^{-iax}}{2(x^2-1)}dx =$$

$$= \lim_{\varepsilon\to 0}\left[\frac{e^{iax}}{2(x^2-1)}dx + \underbrace{\left(\int_0^{1-\varepsilon} + \int_{1+\varepsilon}^{\infty}\right)\frac{e^{-iax}}{2(x^2-1)}dx}_{x\to -x}\right]$$

$$= \lim_{\varepsilon\to 0}\left(\int_{-\infty}^{-1-\varepsilon} + \int_{-1-\varepsilon}^{0} + \int_0^{1-\varepsilon} + \int_{1+\varepsilon}^{\infty}\right)\underbrace{\frac{e^{iax}}{2(x^2-1)}}_{f(x)}dx$$

analytic contunuation to complex arguments:

$$f(x) \to f(z) = \frac{e^{iaz}}{2(z^2-1)}$$

with poles in $z = \pm 1$

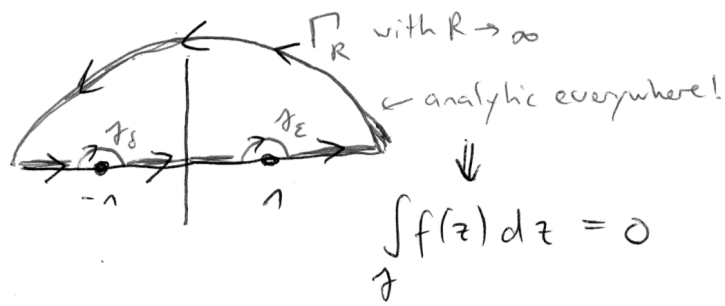
We can write $f(z)$ with partial fractions:

$$f(z) = \frac{e^{iaz}}{2(z^2-1)} = \frac{e^{iaz}}{2}\left[\frac{A}{z+1} + \frac{B}{z-1}\right] = \frac{e^{iaz}}{2}\left[\frac{1}{z-1} - \frac{1}{z+1}\right]$$

Consider now a path integral in the upper half plane $(Im(z) \geq 0)$:

Thus:

$$0 = \lim_{R\to\infty}\int_{\Gamma_R} f(z)dz + \lim_{\delta\to 0}\int_{\gamma_\delta} f(z)dz + \lim_{\varepsilon\to 0}\int_{\gamma_\varepsilon} f(z)dz + I$$



Let us compute the integrals:

$$
\begin{aligned}
z &= Re^{i\varphi} \\
dz &= iRe^{i\phi}d\varphi
\end{aligned}
$$

$$
\begin{aligned}
\lim_{R\to\infty}\int_{\Gamma_R} f(z)dz &= \lim_{R\to\infty}\int_{\Gamma_R}\frac{e^{iaz}}{2(z^2-1)}dz = \lim_{R\to\infty}\int_0^{2\pi}\frac{e^{iaRe^{i\varphi}}}{2(R^2e^{2i\varphi}-1)}iRe^{i\varphi}d\varphi \\
&= \lim_{R\to\infty}\int_0^{2\pi}\underbrace{\frac{iR}{2(R^2\underbrace{e^{2i\varphi}-1)}_{oscillatory}}}_{\to 0}\underbrace{e^{iaRe^{i\varphi}}}_{oscillatory}d\varphi = 0
\end{aligned}
$$

$$
\begin{aligned}
w &\equiv z+1 \\
z &= w-1 \\
dz &= dw \\
w=0:\ \ w &= \delta e^{i\varphi} \qquad\qquad \text{circle around } 0 \\
dw &= i\delta e^{i\varphi}d\varphi
\end{aligned}
$$

$$
\begin{aligned}
\lim_{\delta\to 0}\int_{\gamma_\delta} f(z)dz &= \lim_{\delta\to 0}\frac{1}{4}\left[\int_{\gamma_\delta}\frac{e^{iaz}}{z-1}dz - \int_{\gamma_\delta}\frac{e^{iaz}}{z+1}dz\right] \\
&= \lim_{\delta\to 0}\frac{1}{4}\left[\frac{e^{ia(w-1)}}{w-2}dw - \int_{\gamma_\delta}\frac{e^{ia(w-1)}}{w}dw\right]
\end{aligned}
$$

$$
\begin{aligned}
&\overset{z=-1\Rightarrow w=0}{=} \lim_{\delta\to 0}\frac{e^{-ia}}{4}\left(\underbrace{\frac{\overbrace{e^{ia\delta e^{i\varphi}}}^{\to 1}}{\underbrace{\delta e^{i\varphi}}_{\to 0}-2}\underbrace{i\delta e^{i\varphi}}_{\to 0}d\varphi}_{\to 0} - \int_\pi^0\underbrace{\frac{e^{ia\delta e^{i\varphi}}}{\delta e^{i\varphi}i\delta e^{i\varphi}d\varphi}}_{\to i}\right) \\
&= \frac{i\pi}{4}e^{-ia}
\end{aligned}
$$

$$
\begin{aligned}
u &= z-1 \\
z &= u+1 \\
dz &= du \\
u=0: \quad u &= \varepsilon e^{i\varphi} \\
du &= i\varepsilon e^{i\varphi} d\varphi
\end{aligned}
$$

$$
\begin{aligned}
\lim_{\varepsilon\to 0}\int\limits_{\gamma_\varepsilon} f(z)dz &= \lim_{\varepsilon\to 0}\frac{1}{4}\left[\int\limits_{\gamma_\varepsilon}\frac{e^{ia(u+1}}{u}du - \int\limits_{\gamma_\varepsilon}\frac{e^{ia(u+1}}{u+2}du\right] \\
&\overset{z=1\Rightarrow u=0}{=} \lim_{\varepsilon\to 0}\frac{e^{ia}}{4}\left[\underbrace{\overbrace{\frac{e^{ia\varepsilon e^{i\varphi}}}{\varepsilon e^{i\varphi}}}^{\to 1} i\varepsilon e^{i\varphi}}_{\to i} d\varphi - \underbrace{\int\limits_{\pi}^{0}\underbrace{\overbrace{\frac{e^{ia\varepsilon e^{i\varphi}}}{\underbrace{\varepsilon e^{i\varphi}}_{\to 0}+2}}^{\to 1}\underbrace{i\varepsilon e^{i\varphi}}_{\to 0}}_{\to 0} d\varphi}\right] \\
&= \frac{-i\pi}{4}e^{ia}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow 0 &= 0 + \frac{i\pi}{4}e^{-ia} - \frac{i\pi}{4}e^{ia} + I \\
I &= \frac{i\pi}{4}\left[e^{ia} - e^{-ia}\right] = -\frac{\pi}{2}\sin a
\end{aligned}
$$